# DECAY FOR THE WAVE AND SCHRÖDINGER EVOLUTIONS ON MANIFOLDS WITH CONICAL ENDS, PART I

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a compact imbedded Riemannian manifold of dimension $d \geq 1$ and define the $(d+1)$-dimensional Riemannian manifold $\mathcal{M} := \{(x, r(x)\omega) \,:\, x \in \mathbb{R}, \omega \in \Omega\}$ with $r > 0$ and smooth, and the natural metric $ds^2 = (1+r'(x)^2)dx^2 + r^2(x)ds^2_\Omega$. We require that $\mathcal{M}$ has conical ends: $r(x) = |x| + O(x^{-1})$ as $x \to \pm\infty$. The Hamiltonian flow on such manifolds always exhibits trapping. Dispersive estimates for the Schrödinger evolution $e^{it\Delta_\mathcal{M}}$ and the wave evolution $e^{it\sqrt{-\Delta_\mathcal{M}}}$ are obtained for data of the form $f(x,\omega) = Y_n(\omega)u(x)$ where $Y_n$ are eigenfunctions of $\Delta_\Omega$. This paper treats the case $d = 1$, $Y_0 = 1$. In Part II of this paper we provide details for all cases $d + n > 1$. Our method combines two main ingredients:
(A) a detailed scattering analysis of Schrödinger operators of the form $-\partial^2_\xi + V(\xi)$ on the line where $V(\xi)$ has inverse square behavior at infinity
(B) estimation of oscillatory integrals by (non)stationary phase.

## 1. Introduction

It is well-known that the free Schrödinger evolution on $\mathbb{R}^{n+1}$ satisfies the dispersive bound

$$\|e^{it\Delta}f\|_\infty \lesssim |t|^{-\frac{n}{2}}\|f\|_1 \tag{1.1}$$

where $\Delta$ denotes the Laplacean in $\mathbb{R}^n$. Similarly, solutions to the wave equation

$$\Box u = 0, \quad u(0) = u_0,\ \partial_t u(0) = u_1$$

in $\mathbb{R}^{n+1}$ satisfy

$$\begin{aligned} \|u(t,\cdot)\|_\infty &\lesssim t^{-\frac{n-1}{2}}\Big(\|u_0\|_{\dot W^{\frac{n+1}{2},1}} + \|u_1\|_{\dot W^{\frac{n-1}{2},1}}\Big) \\ \|u(t,\cdot)\|_\infty &\lesssim t^{-\frac{n-1}{2}}\Big(\|u_0\|_{\dot B^{\frac{n+1}{2}}_{1,1}} + \|u_1\|_{\dot B^{\frac{n-1}{2}}_{1,1}}\Big) \end{aligned} \tag{1.2}$$

in odd and even dimensions, respectively. Another instance of such decay bounds are the global Strichartz estimates

$$\|e^{it\Delta}f\|_{L^{2+\frac{4}{n}}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{1.3}$$

and mixed-norm variants thereof as well as the corresponding versions for the wave equation.

In this paper we establish a decay estimate (valid for all $t$), similar to (1.1), for the Schrödinger and wave evolution on a class of non-compact manifolds which exhibit trapping of the Hamiltonian flow. There has been much activity around

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establishing dispersive and Strichartz estimates for more general operators, namely for Schrödinger operators of the form $H = -\Delta + V$ with a decaying potential $V$ or even more general perturbations. The seminal papers here are Rauch[15], Jensen-Kato[13], and Jorneé-Soffer-Sogge[14]. We refer the reader to the survey [19] for more recent references in this area.

Around the same time as [14], Bourgain[3] found Strichartz estimates on the torus. This is remarkable, as compact manifolds do not exhibit dispersion as in (1.1) which was always considered a key ingredient of the $T^*T$ argument leading to (1.3). The theme of Strichartz estimates on manifolds (both local and global in time) was then developed further in several important papers, see Smith-Sogge[20], Staffilani-Tataru[21], Burq-Gerard-Tzvetkov[4], [5], Hassel-Tao-Wunsch[11], [12], Robbiano-Zuily[16], and Tataru[22]. Gerard[9] reviews some of the recent work in this field.

A recurring theme in this area is the importance of periodic geodesics for Strichartz estimates. In fact, it is well-known that the presence of periodic geodesics can lead to a loss of derivatives in the Strichartz bounds. The intuition here is that initial data that are highly localized around a periodic geodesic and possess high momentum traveling around this geodesic will lead to so-called meta-stable states in the Schrödinger evolution *provided the geodesic is stable* as for example on spheres. Metastable states remain "coherent" for a long time, which amounts to absence of dispersion during that time, see for example [9] (in the classical approximation, dispersive estimates are governed by the Newtonian scattering trajectories — classically speaking, periodic geodesics are states that do not scatter).

For this reason, many authors have imposed explicit non-trapping conditions, see [20], [11], [12], [17]. The relevance of this condition lies with the construction of a parametrix, which always involves solving for suitable bi-characteristics. On manifolds these bi-characteristics are governed by the geodesics flow in the cotangent bundle - hence the relevance of periodic geodesics.

There is a large body of work on the so-called Kato smoothing estimates where this non-trapping condition also features prominently, see for example Craig-Kappeler-Strauss[6], Doi[8], and Rodnianski-Tao[17].

We now define the class of asymptotically conical manifolds $\mathcal{M}$ that we shall be working with.

**Definition 1.1.** *Let $\Omega \subset \mathbb{R}^N$ with metric $ds_\Omega^2$ be a $d$-dimensional compact imbedded Riemannian manifold and define the $(d+1)$-dimensional manifold*

$$\mathcal{M} := \{(x, r(x)\omega) \mid x \in \mathbb{R},\ \omega \in \Omega\}, \quad ds^2 = r^2(x) ds_\Omega^2 + (1 + r'(x)^2) dx^2$$

*where $r \in C^\infty(\mathbb{R})$ and $\inf_x r(x) > 0$. We say that there is a* conical end *at the right (or left) if*

$$r(x) = |x|\,(1 + h(x)), \quad h^{(k)}(x) = O(x^{-2-k}) \quad \forall\, k \geq 0 \tag{1.4}$$

*as $x \to \infty$ ($x \to -\infty$).*

Of course we can consider cones with arbitrary opening angles here but this adds nothing of substance. Furthemore, the regularity assumption can be relaxed to finitely many derivatives, but we do not comment on this issue any further. With $\Omega = S^1$ the manifold $\mathcal{M}$ reduces to a surface of revolution

$$\mathcal{S} = \{(x, r(x)\cos\theta, r(x)\sin\theta) : -\infty < x < \infty, 0 \leq \theta \leq 2\pi\}$$

with the metric $ds^2 = r^2(x) d\theta^2 + (1 + r'(x)^2) dx^2$. It has a periodic geodesic at all local extrema of $r$. An example of such a manifold is given by the one-sheeted

hyperboloid: $\Omega = S^1$ and $r(x) = \sqrt{1+|x|^2} =: \langle x \rangle$. If $d \geq 2$, the entire Hamiltonian flow on $\mathcal{M}$ is trapped on the set $(x_0, r(x_0)\Omega)$ when $r'(x_0) = 0$.

In what follows, $\{Y_n, \mu_n\}_{n=0}^{\infty}$ denote the $L^2$-normalized eigenfunctions and eigenvalues, respectively, of $-\Delta_\Omega$. In other words, $-\Delta_\Omega Y_n = \mu_n^2 Y_n$ where $0 = \mu_0^2 < \mu_1^2 \leq \mu_2^2 \leq \dots$

**Theorem 1.2.** *Let $\mathcal{M}$ be asymptotically conical at both ends in the sense of Definition 1.1 with $d \geq 1$ arbitrary. Then for all $t > 0$ and all $n \geq 0$,*

$$\|e^{it\Delta_{\mathcal{M}}} Y_n f\|_{L^\infty(\mathcal{M})} \leq C(n, \mathcal{M})\, t^{-\frac{d+1}{2}} \|f\|_{L^1(\mathcal{M})} \tag{1.5}$$

$$\|e^{\pm it\sqrt{-\Delta_{\mathcal{M}}}} Y_n f\|_{L^\infty(\mathcal{M})} \leq C(n, \mathcal{M})\, t^{-\frac{d}{2}} \Big( \|f'\|_{L^1(\mathcal{M})} + \|f\|_{L^1(\mathcal{M})} \Big) \tag{1.6}$$

*provided $f = f(x)$ does not depend on $\omega$.*

We remark that in the flat case, i.e., $r = \text{const} = 1$ the evolutions factor into those on $\Omega$ and $\mathbb{R}$ and the dispersive rates are of course the same as on $\mathbb{R}$. As for the wave equation, (1.6) gives the natural estimate for $\cos(t\sqrt{-\Delta_{\mathcal{M}}})$ — the number of derivatives appearing on the right-hand side agrees with that in (1.2) when $n = 1$ since $f$ really sees the evolution along a one-dimensional generator the "missing" angular derivatives being hidden in $C(n, \mathcal{M})$. For $\frac{\sin(t\sqrt{-\Delta_{\mathcal{M}}})}{\sqrt{-\Delta_{\mathcal{M}}}}$ one can prove the stronger bound which only requires $L^1$ data, but we do not elaborate on this here.

In this paper we only prove the case $d = 1, n = 0$. In Part II we consider the general case. It turns out that the all cases subsumed in $d + n > 1$ follow very much the same scheme whereas $d + n = 1$ has some separate features. This is to be expected, as for $N = 2$ the dispersive estimates for $-\Delta_{\mathbb{R}^N} + V$ are quite different from those in $\mathbb{R}^N$ with $N \geq 3$, compare [18] to [14]. This is due to the logarithmic singularity of $(-\Delta_{\mathbb{R}^2} - z)^{-1}$ at $z = 0$ as compared to the boundedness of the resolvent when $N \geq 3$. Not surprisingly, the logarithmic issues reappear in Part I but not in Part II of this series.

We now briefly describe the main ideas behind the proofs of Theorem 1.2. First, using arc-length coordinates $\xi$ on $\mathcal{M}$ and after multiplying by the weight $r^{\frac{d}{2}}(\xi)$, we reduce matters to the Schrödinger operator

$$\mathcal{H}_{d,n} := -\partial_\xi^2 + \frac{\mu_n^2}{r^2(\xi)} + V_1(\xi) =: -\partial_\xi^2 + V(\xi)$$

on $\mathbb{R}_\xi$. Here $V_1(\xi)$ is a smooth potential that behaves like $\frac{1}{4}d(d-2)\xi^{-2}$ as $\xi \to \pm\infty$. If $d = 1, n = 0$, then $V(\xi) \sim -\frac{1}{4\xi^2}$ as $\xi \to \infty$ (it is therefore an attractive potential), whereas for $d + n > 1$ the potential $V$ becomes repulsive (in fact, very much so as $n$ and $d$ increase). On the one hand, this difference accounts for the separate treatment of $d + n = 1$ here as opposed to part II. On the other hand, since

$$V(\xi) = \big[2\mu_n^2 + d(d-2)/4\big]\xi^{-2} + O(\xi^{-3}) \qquad \text{as } |\xi| \to \infty,$$

with a positive leading term when $d+n > 1$, it is reasonable that the cases $d+n > 1$ can be treated simultaneously.

In order to prove our theorems, we express the resolvent kernel as

$$(\mathcal{H}_{d,n} - (\lambda^2 + i0))^{-1}(\xi, \xi') = \frac{f_+(\xi,\lambda) f_-(\xi',\lambda)}{W(\lambda)}$$

when $\xi > \xi'$. Here $f_\pm$ are the usual Jost solutions for $\mathcal{H}_{d,n}$ at energy $\lambda^2$:

$$\mathcal{H}_{d,n} f_\pm(\cdot,\lambda) = \lambda^2 f_\pm(\cdot,\lambda), \qquad f_\pm(\cdot,\lambda) \sim e^{i\xi\lambda} \text{ as } \xi \to \pm\infty$$

and

$$W(\lambda) = W(f_-(\cdot,\lambda), f_+(\cdot,\lambda))$$

is their Wronskian.

Let us now briefly recall what is know about the existence of the Jost solutions and the asymptotic behavior of $W(\lambda)$ for general operators $\mathcal{H} = -\partial_\xi^2 + V$, see for example Deift-Trubowitz [7] for these elementary facts of scattering theory: for potentials $V(\xi)$ satisfying $\langle\xi\rangle V(\xi) \in L^1(\mathbb{R})$ the Jost solutions exist and are continuous in $\lambda \in \mathbb{R}$ (in fact, they are continuous in $\lambda \neq 0$ under the weaker condition $V \in L^1$). Moreover, $W(\lambda) \sim 2i\lambda$ as $\lambda \to \infty$ and either $W(0) \neq 0$ or $W(\lambda) \sim c\lambda$ as $\lambda \to 0$. The former case is said to be *nonresonant* whereas the latter is *resonant*; it occurs exactly if there is globally bounded nonzero solution to $\mathcal{H}f = 0$. In the nonresonant case, $f \sim 1$ as $\xi \to \infty$ then necessarily implies that $f(\xi)$ grows linearly in $\xi$ as $\xi \to -\infty$.

In the case of an inverse square potential the behavior of $f_\pm(\cdot,\lambda)$ and thus also of $W(\lambda)$ as $\lambda \to 0$ is radically different. Assuming for simplicity that the leading order asymptotic behavior of $V(\xi)$ is the same as $\xi \to \pm\infty$ (as it is here) we single out two possible scenarios which emerge from our analysis: first, suppose that

$$V(\xi) = (\nu^2 - \frac{1}{4})\xi^{-2} + O(\xi^{-3}) \ \text{ as } \ \xi \to \infty$$

where $\nu > 0$ (the case $\nu = 0$ differing by logarithmic corrections). Then either $W(\lambda) \sim c\lambda^{1-2\nu}$ or $W(\lambda) \sim c\lambda^\sigma$ for some $\sigma < 1 - 2\nu$ as $\lambda \to 0$. Loosely speaking, the former can be viewed as an analogue of the *nonresonant* case from the usual scattering theory whereas the latter is the *resonant* case. The resonant case is characterized by the existence of a nonzero solution $u$ of $\mathcal{H}u = 0$ with asymptotic behavior $\xi^{\frac{1}{2}-\nu}$ as $\xi \to \infty$ and $c\,|\xi|^{\frac{1}{2}-\nu}$ as $\xi \to -\infty$ where $c \neq 0$. Note that in the special case $\nu = \frac{1}{2}$, which puts us back in the $\langle\xi\rangle V \in L^1$ scenario, this is exactly the standard characterization of a zero energy resonance: there exists a nontrivial globally bounded zero energy solution. In the resonant case one might expect $\sigma = 1$, but our analysis does not yield that conclusion.

To conclude this introduction, let us recall the well-known heuristic principle that the behavior of the spectral measure close to zero energy is the decisive fact for the long term behavior of any wave evolution. Indeed, with $E$ being the spectral resolution of $\mathcal{H}_{d,n}$,

$$e^{it\mathcal{H}_{d,n}} = \int_0^\infty e^{it\lambda} E(d\lambda)$$

Thus, decay of this Fourier transform as $t \to \infty$ is reflected most strongly by the behavior of $E(d\lambda)$ around $\lambda = 0$. This of course explains the importance of analyzing $W(\lambda)$ close to $\lambda = 0$.

We now describe the proof method in more detail.

## 2. The basic setup

The Laplace-Beltrami operator on $\mathcal{M}$ where the base $\Omega$ is of dimension $d \geq 1$, is

$$\Delta_{\mathcal{M}} = \frac{1}{r^d(x)\sqrt{1 + r'(x)^2}}\,\partial_x \left( \frac{r^d(x)}{\sqrt{1 + r'(x)^2}} \partial_x \right) + \frac{1}{r^2(x)} \Delta_\Omega \tag{2.1}$$

We switch to arclength parametrization. Thus, let

$$\xi(x) = \int_0^x \sqrt{1+r'(y)^2}\, dy.$$

Then (2.1) can be written as

$$\Delta_{\mathcal{M}} = \frac{1}{r^d(\xi)}\partial_\xi(r^d(\xi)\partial_\xi) + \frac{1}{r^2(\xi)}\Delta_\Omega \tag{2.2}$$

where we have abused notation: $r(\xi)$ instead of $r(x(\xi))$. Setting $\rho(\xi) := \frac{d}{2}\frac{\dot r(\xi)}{r(\xi)}$ yields

$$\Delta_{\mathcal{M}}\, y(\xi,\omega) = \partial_\xi^2 y + 2\rho\partial_\xi y + \frac{1}{r^2}\Delta_\Omega y. \tag{2.3}$$

We remove the first order term in (2.3) by setting

$$y(\xi,\omega) = r(\xi)^{-\frac{d}{2}} u(\xi,\omega). \tag{2.4}$$

Then

$$\Delta_{\mathcal{M}} y = \partial_\xi^2 y + 2\rho\partial_\xi y + \frac{1}{r^2}\Delta_\Omega y = r^{-d/2}[-\mathcal{H}u + \frac{1}{r^2}\Delta_\Omega u] \tag{2.5}$$

with

$$\mathcal{H} = -\partial_\xi^2 + V, \quad V(\xi) = \rho^2(\xi) + \dot\rho(\xi). \tag{2.6}$$

Note that the Schrödinger operator $\mathcal{H}$ can be factorized as

$$\mathcal{H} = \mathcal{L}^*\mathcal{L}, \quad \mathcal{L} = -\frac{d}{d\xi} + \rho \tag{2.7}$$

In particular, $\mathcal{H}$ has no negative spectrum. In terms of the Schrödinger evolution,

$$e^{-it\Delta_{\mathcal{M}}} f = r^{-\frac{d}{2}} e^{it\mathcal{H}} r^{\frac{d}{2}} f \quad \forall\, f = f(\xi)$$

and the same for the wave equation. In particular, any estimate of the form

$$\|e^{-it\Delta_{\mathcal{M}}} f\|_{L^\infty(\mathcal{M})} \le C t^{-\alpha} \|f\|_{L^1(\mathcal{M})} \quad \forall\, t>0,\ f = f(\xi)$$

with arbitrary $\alpha \ge 0$ and some constant $C$ that does not depend on $t$, is equivalent to one of the form

$$\big\|r^{-\frac{d}{2}} e^{it\mathcal{H}} r^{-\frac{d}{2}} u\big\|_{L^\infty(\mathbb{R})} \le C'\, t^{-\alpha} \|u\|_{L^1(\mathbb{R})} \quad \forall\, t>0,\ u = u(\xi) \tag{2.8}$$

with a possibly different constant $C'$. Here we absorbed the weight from the volume element $dv_{\mathcal{M}} = r^d d\xi dv_\Omega$ arising in the $L^1(\mathcal{M})$ norm into the left-hand side of (2.8). An analogous reduction is of course valid for the wave evolution. As usual, the functional calculus applied to (2.8) yields

$$e^{it\mathcal{H}} = \int_0^\infty e^{it\lambda} E(d\lambda)$$

where $E(d\lambda)$ is the spectral resolution of $\mathcal{H}$. The point is that there is an "explicit expression" for $E(d\lambda)$:

$$E(d\lambda^2)(\xi,\xi') = 2\lambda\Big\{\mathrm{Im}\Big[\frac{f_+(\xi,\lambda)f_-(\xi',\lambda)}{W(\lambda)}\Big]\chi_{[\xi>\xi']} + \mathrm{Im}\Big[\frac{f_-(\xi,\lambda)f_+(\xi',\lambda)}{W(\lambda)}\Big]\chi_{[\xi<\xi']}\Big\}\, d\lambda$$

where

$$W(\lambda) := W(f_-(\cdot,\lambda), f_+(\cdot,\lambda)) = f_+'(\cdot,\lambda)f_-(\cdot,\lambda) - f_-'(\cdot,\lambda)f_+(\cdot,\lambda)$$

is the Wronskian of the solutions $f_\pm(\cdot,\lambda)$ of the following ordinary differential equation

$$\begin{aligned} \mathcal{H} f_\pm(\xi,\lambda) &= -f_\pm''(\xi,\lambda) + V(\xi) f_\pm(\xi,\lambda) = \lambda^2 f_\pm(\xi,\lambda) \\ f_\pm(\xi,\lambda) &\sim e^{\pm i\lambda\xi} \qquad \text{as } \xi \to \pm\infty \end{aligned} \tag{2.9}$$

provided $\lambda \neq 0$. The functions $f_\pm$ are called the *Jost solutions* and it is a standard fact that these solutions exist because of the decay of $V$ which turns out to be

$$|V(\xi)| \lesssim \langle \xi \rangle^{-2}.$$

To establish this, as well as an important refinement thereof, we start with the following elementary consequence of Definition 1.1.

**Definition 2.1.** *In what follows, a term $O(x^{-\gamma})$ is said to* behave like a symbol *if $|\partial_x^\ell O(x^{-\gamma})| \lesssim x^{-\gamma-\ell}$ as $x \to \infty$ for all $\ell \geq 1$.*

Furthermore, we shall assume henceforth that both ends of $\mathcal{M}$ are conical, i.e., (1.4) holds.

**Lemma 2.2.** *With suitable constants $c_\infty, \tilde{c}_\infty$, and as $x \to \infty$*

$$\xi(x) = \sqrt{2}\, x + c_\infty + O(x^{-1}) \tag{2.10}$$

*as well as*

$$r(\xi) = \frac{1}{\sqrt{2}} \xi \left(1 - \frac{c_\infty}{\xi} + O(\xi^{-2})\right) \tag{2.11}$$

*as $\xi \to \infty$. Moreover, the $O$-terms behave like symbols.*

*Proof.* We plug $r(x) = x(1 + O(x^{-2}))$ and thus $r'(x) = 1 + O(x^{-2})$ into the expression for $\xi$, i.e.,

$$\begin{aligned} \xi(x) &= \int_0^x \sqrt{2 + O(\langle y \rangle^{-2})}\, dy = \sqrt{2}\, x + \int_0^x O(\langle y \rangle^{-2})\, dy \\ &= \sqrt{2}\, x + \int_0^\infty O(\langle y \rangle^{-2})\, dy + O(x^{-1}) = \sqrt{2}\, x + c_\infty + O(x^{-1}) \end{aligned}$$

Hence,

$$r(x) = x + O(x^{-1}) = 2^{-\frac{1}{2}} (\xi - c_\infty) + O(\xi^{-1})$$

as claimed. The symbol behavior follows from the fact that the errors in Definition 1.1 also behave like symbols. □

As a corollary, we obtain

**Corollary 2.3.** *The potential $V$ from (2.6) has the form*

$$V(\xi) = \Big(\frac{d^2}{4} - \frac{d}{2}\Big)\xi^{-2} + O(\xi^{-3}) \qquad \text{as } \xi \to \infty \tag{2.12}$$

*where $O(\xi^{-3})$ behaves like a symbol.*

*Proof.* Simply observe that at a conical end, $\rho = \frac{d}{2}\frac{\dot{r}}{r} = \frac{d}{2}\xi^{-1}(1 + O(\xi^{-1}))$ as $\xi \to \infty$. Hence,

$$V(\xi) = \dot{\rho}(\xi) + \rho^2(\xi) = \frac{1}{4} d(d-2)\xi^{-2} + O(\xi^{-3}) \;\text{ as }\; \xi \to \infty$$

as claimed. The behavior of the $O(\cdot)$ term follows from the fact that the $O(\cdot)$ in Lemma 2.2 are of symbol type. □

From (2.9), $f_\pm(\cdot,\lambda)$ are solutions of the *Volterra integral equations*

$$f_+(\xi,\lambda) = e^{i\lambda\xi} + \int_\xi^\infty \frac{\sin(\lambda(\eta-\xi))}{\lambda} V(\eta) f_+(\eta,\lambda)\,d\eta \tag{2.13}$$

and similarly for $f_-$. For the convenience of the reader, we now recall how to solve Volterra integral equations in general. Thus, consider

$$(*)\ \ f(x) = g(x) + \int_x^\infty K(x,s)f(s)ds,$$

or

$$(**)\ \ f(x) = g(x) + \int_a^x K(x,s)f(s)ds,$$

with some $g(x) \in L^\infty$ and $a \in \mathbb{R}$. As usual, one solves them by an iteration procedure which requires finding a suitable convergent majorant for the resulting series expansion.

**Lemma 2.4.** *Let $a \in \mathbb{R}$ and $g(x) \in L^\infty(a,\infty)$. Let*

$$\mu := \int_a^\infty \sup_{a<x<s} |K(x,s)|\,ds < \infty$$

*Then there exists a unique solution to $(*)$ given by*

$$f(x) = g(x) + \sum_{n=1}^\infty \int_a^\infty \dots \int_a^\infty \prod_{i=1}^n \chi_{[x_{i-1}<x_i]} K(x_{i-1},x_i)\ g(x_n)\,dx_n \dots dx_1. \tag{2.14}$$

*with $x_0 := x$. Furthermore, one has the bound*

$$\|f\|_{L^\infty(a,\infty)} \le e^\mu \|g\|_{L^\infty(a,\infty)},$$

*and an analogue statement holds for $(**)$.*

*Proof.* We only prove the lemma for $(*)$ since the proof for $(**)$ is almost identical. The idea is simply to show that the infinite Volterra iteration (2.14) for $(*)$ converges. To this end, define

$$K_0(s) := \sup_{a<x<s} |K(x,s)|$$

Then

$$\begin{aligned}
&\Big| \int_a^\infty \dots \int_a^\infty \prod_{i=1}^n \chi_{[x_{i-1}<x_i]} K(x_{i-1},x_i)\ g(x_n)\,dx_n \dots dx_1 \Big| \\
&\le \int_a^\infty \dots \int_a^\infty \prod_{i=1}^n \chi_{[x_{i-1}<x_i]} K_0(x_i)\ |g(x_n)|\,dx_n \dots dx_1 \\
&= \|g\|_{L^\infty(a,\infty)} \frac{1}{n!} \int_a^\infty \dots \int_a^\infty \prod_{i=1}^n K_0(x_i)\,dx_n \dots dx_1 \\
&= \frac{1}{n!} \|g\|_{L^\infty(a,\infty)} \Big( \int_a^\infty K_0(s)\,ds \Big)^n
\end{aligned}$$

Hence, the series in (2.14) converges absolutely and uniformly in $x > a$ with the uniform upper bound

$$\|g\|_{L^\infty(a,\infty)} \sum_{n=0}^\infty \frac{1}{n!}\mu^n = e^\mu \|g\|_{L^\infty(a,\infty)}$$

as claimed. □

It is now clear that (2.13) admits a solution for every $\lambda \neq 0$. At $\lambda = 0$, we need to replace (2.13) with

$$f_+(\xi,0) = 1 + \int_\xi^\infty (\eta - \xi) V(\eta) f_+(\eta,0)\, d\eta$$

If $d \neq 2$, then this integral equation has no meaning due to the $\eta^{-2}$ decay of $V(\eta)$, see (2.12). Moreover, the zero energy solutions of $\mathcal{H}u = 0$ are given by

$$\begin{aligned} u_0(\xi) &= r^{\frac{d}{2}}(\xi), \\ u_1(\xi) &= r^{\frac{d}{2}}(\xi) \int_0^\xi r^{-d}(\eta)\, d\eta, \end{aligned} \tag{2.15}$$

see (3.1) and (2.4). Since no linear combination of these functions can be made asymptotically constant when $d \neq 2$, it follows that (2.9) itself has no meaning at $\lambda = 0$. Note, however, that for $d = 2$

$$r(\xi) \int_\xi^\infty r^{-2}(\eta)\, d\eta$$

is asymptotically constant at a conical end as $\xi \to \infty$ which is in agreement with the fact that for $d = 2$ the potential $V$ decays like an inverse cubic.

In view of this discussion, we have reduced the decay estimates for the Schrödinger equation to the following oscillatory integral bounds:

$$\begin{aligned} &\sup_{\xi > \xi'} r^{-\frac{d}{2}}(\xi) r^{-\frac{d}{2}}(\xi') \Big| \int_0^\infty e^{it\lambda^2} \lambda \,\mathrm{Im}\left[\frac{f_+(\xi,\lambda) f_-(\xi',\lambda)}{W(\lambda)}\right] d\lambda \Big| \\ &+ \sup_{\xi < \xi'} r^{-\frac{d}{2}}(\xi) r^{-\frac{d}{2}}(\xi') \Big| \int_0^\infty e^{it\lambda^2} \lambda \,\mathrm{Im}\left[\frac{f_+(\xi',\lambda) f_-(\xi,\lambda)}{W(\lambda)}\right] d\lambda \Big| \lesssim t^{-(d+1)/2} \end{aligned} \tag{2.16}$$

For the wave-equation, the reduction takes the form

$$\begin{aligned} &\Big| \int_{-\infty}^\xi r^{-\frac{d}{2}}(\xi) r^{-\frac{d}{2}}(\xi') \int_0^\infty e^{it\lambda}\, \lambda \,\mathrm{Im}\left[\frac{f_+(\xi,\lambda) f_-(\xi',\lambda)}{W(\lambda)}\right] d\lambda\, \phi(\xi')\, d\xi' \Big| \\ &+ \Big| \int_\xi^\infty r^{-\frac{d}{2}}(\xi) r^{-\frac{d}{2}}(\xi') \int_0^\infty e^{it\lambda}\, \lambda \mathrm{Im}\left[\frac{f_+(\xi',\lambda) f_-(\xi,\lambda)}{W(\lambda)}\right] d\lambda\, \phi(\xi')\, d\xi' \Big| \\ &\lesssim t^{-d/2} \int (|\phi'(\eta)| + |\phi(\eta)|)\, d\eta \end{aligned} \tag{2.17}$$

uniformly in $\xi$.

## 3. The scattering theory for $d = 1, n = 0$

The goal of this section is to obtain a sufficiently accurate representation of $f_\pm(\cdot,\lambda)$ in (2.16) and (2.17). We remark that using (2.2), one obtains two $\omega$ independent harmonic functions on $\mathcal{M}$:

$$y_0(\xi) = 1, \quad y_1(\xi) = \int_0^\xi r^{-1}(\xi')\, d\xi' \tag{3.1}$$

At a conical end, $y_1(\xi) = \sqrt{2}\log\xi + O(1)$, cf. Lemma 2.2. The related functions $u_0 = r^{\frac{1}{2}}$ and $u_1 = r^{\frac{1}{2}} y_1$ from (2.15) are zero-energy solutions of $\mathcal{H}$, see (2.6)

and (2.7). Their asymptotics are as follows (assuming throughout that $\mathcal{M}$ is conical at the ends):

**Lemma 3.1.** *As $\xi \to \infty$,*

$$\begin{aligned} u_0(\xi) &= 2^{-1/4}\xi^{1/2}\Big(1-\frac{c_\infty}{2\xi}+O(\xi^{-2})\Big) \\ u_1(\xi) &= 2^{1/4}\xi^{1/2}\Big(1-\frac{c_\infty}{2\xi}+O(\xi^{-2})\Big)\Big(\log\xi+c_2+O(\xi^{-1})\Big). \end{aligned} \tag{3.2}$$

*Here $c_2$ is some constant and the $O$-terms behave like symbols under differentiation in $\xi$.*

*Proof.* The expressions for $u_0$ are an immediate consequence of Lemma 2.2. Simply compute

$$\begin{aligned} \int_0^\xi r^{-1}(\eta)\,d\eta &= \int_0^\xi \sqrt{2}\,\langle\eta\rangle^{-1}\big(1+c_\infty\langle\eta\rangle^{-1}+O(\langle\eta\rangle^{-2})\big)\,d\eta \\ &= \sqrt{2}\,(\log\xi+c_2)+O(\xi^{-1}) \qquad \text{as } \xi\to\infty. \end{aligned}$$

Thus,

$$\begin{aligned} u_1(\xi) &= \sqrt{r(\xi)}\int_0^\xi r^{-1}(\eta)\,d\eta \\ &= 2^{1/4}\xi^{1/2}\left(1-\frac{c_\infty}{2\xi}+O(\xi^{-2})\right)\big(\log\xi+c_2+O(\xi^{-1})\big) \qquad \text{as } \xi\to\infty. \end{aligned}$$

To symbol character of the $O(\cdot)$ terms here follows from the fact that it was assumed in Definition 1.1. □

We now perturb the zero energy solutions relative to the energy. For small energies and in the region $|\xi\lambda| \ll 1$, this produces a useful approximation to the exact solutions.

**Lemma 3.2.** *For any $\lambda \in \mathbb{R}$, define*

$$u_j(\xi,\lambda) := u_j(\xi) + \lambda^2\int_0^\xi [u_1(\xi)u_0(\eta)-u_1(\eta)u_0(\xi)]u_j(\eta,\lambda)\,d\eta \tag{3.3}$$

*where $j=0,1$. Then $\mathcal{H}u_j(\cdot,\lambda)=\lambda^2 u_j(\cdot,\lambda)$ with $u_j(\cdot,0)=u_j(\cdot)$, for $j=0,1$ and*

$$W(u_0(\cdot,\lambda),u_1(\cdot,\lambda)) = 1 \tag{3.4}$$

*for all $\lambda$.*

*Proof.* First, one checks that $W(u_0,u_1)=1$. This yields $\mathcal{H}u_j(\cdot,\lambda)=\lambda^2 u_j(\cdot,\lambda)$ since $\mathcal{H}u_j=0$ for $j=0,1$. Second, $u_j(0,\lambda)=u_j(0)$ and $u_j'(0,\lambda)=u_j'(0)$ for $j=0,1$. Hence $W(u_0(\cdot,\lambda),u_1(\cdot,\lambda)) = u_1'(0)u_0(0)-u_1(0)u_0'(0)=1$. □

As an immediate corollary we have the following statement.

**Corollary 3.3.** *There exist $a_+(\lambda)$, $a_-(\lambda)$, $b_+(\lambda)$ and $b_-(\lambda)$ such that with $f_\pm(\cdot,\lambda)$ as in (2.9), one has for any $\lambda\neq 0$*

$$\begin{aligned} f_+(\xi,\lambda) &= a_+(\lambda)u_0(\xi,\lambda)+b_+(\lambda)u_1(\xi,\lambda) \\ f_-(\xi,\lambda) &= a_-(\lambda)u_0(\xi,\lambda)+b_-(\lambda)u_1(\xi,\lambda). \end{aligned} \tag{3.5}$$

*Furthermore $a_\pm(\lambda)=W(f_\pm(\cdot,\lambda),u_1(\cdot,\lambda))$, $b_\pm(\lambda)=-W(f_\pm(\cdot,\lambda),u_0(\cdot,\lambda))$, and*

$$W(\lambda) := W(f_-(\cdot,\lambda),f_+(\cdot,\lambda)) = a_-(\lambda)b_+(\lambda)-a_+(\lambda)b_-(\lambda). \tag{3.6}$$

*Moreover, if $\mathcal{M}$ is symmetric, then $a_-(\lambda) = a_+(\lambda)$ and $b_-(\lambda) = -b_+(\lambda)$.*

*Proof.* The Wronskian relations for $a_\pm$, $b_\pm$ follow immediately from (3.4). The formula for $W(\lambda)$ also follows by plugging (3.5) into (3.6). In the symmetric case, i.e., assuming $r(x) = r(-x)$ one also has $r(\xi) = r(-\xi)$. In particular, this implies that $f_-(-\xi,\lambda) = f_+(\xi,\lambda)$ and $u_0(-\xi) = u_0(\xi)$ as well as $u_1(-\xi) = -u_1(\xi)$. Thus,

$$\begin{aligned} a_-(\lambda) &= W(f_-(\cdot,\lambda), u_1(\cdot,\lambda)) = -W(f_-(-\cdot,\lambda), u_1(-\cdot,\lambda)) \\ &= W(f_+(\cdot,\lambda), u_1(\cdot,\lambda)) = a_+(\lambda) \\ b_-(\lambda) &= -W(f_-(\cdot,\lambda), u_0(\cdot,\lambda)) = W(f_-(-\cdot,\lambda), u_0(-\cdot,\lambda)) \\ &= W(f_+(\cdot,\lambda), u_0(\cdot,\lambda)) = -b_+(\lambda) \end{aligned}$$

as claimed. □

3.1. **The analysis of $f_+(\cdot,\lambda)$ at a conical end, $d = 1$.** By Corollary 2.3,

$$V(\xi) = -\frac{1}{4\xi^2} + V_1(\xi), \qquad \xi \to \infty \tag{3.7}$$

where $|V_1(\xi)| \lesssim |\xi|^{-3}$. Moreover, $|V_1^{(k)}(\xi)| \lesssim |\xi|^{-3-k}$ for $\xi > 1$.

**Lemma 3.4.** *Let*

$$\mathcal{H}_0 := -\partial_\xi^2 - \frac{1}{4\xi^2}$$

*For any $\lambda > 0$ the problem*

$$\begin{aligned} \mathcal{H}_0 f_0(\cdot,\lambda) &= \lambda^2 f_0(\cdot,\lambda), \\ f_0(\xi,\lambda) &\sim e^{i\xi\lambda} \quad \textit{as} \quad \xi \to \infty \end{aligned}$$

*has a unique solution on $\xi > 0$. It is given by*

$$f_0(\xi,\lambda) = \sqrt{\frac{\pi}{2}}\, e^{i\pi/4} \sqrt{\xi\lambda}\, H_0^{(+)}(\xi\lambda). \tag{3.8}$$

*Here $H_0^{(+)}(z) = J_0(z) + iY_0(z)$ is the Hankel function of order zero.*

*Proof.* It is well-known, see Abramowitz-Stegun[1], that the ordinary differential equation

$$w''(z) + \left(\lambda^2 + \frac{1}{4z^2}\right) W(z) = 0$$

has a fundamental system of solutions $\sqrt{z}\, J_0(\lambda z)$, $\sqrt{z}\, Y_0(\lambda z)$ or equivalently,

$$\sqrt{z}\, H_0^{(+)}(\lambda z), \qquad \sqrt{z}\, H_0^{(-)}(\lambda z).$$

Recall the asymptotic relations

$$\begin{aligned} H_0^{(+)}(x) &\sim \sqrt{\frac{2}{\pi x}}\, e^{i(x-\frac{\pi}{4})} \qquad \text{as } x \to +\infty \\ H_0^{(-)}(x) &\sim \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\frac{\pi}{4})} \qquad \text{as } x \to +\infty. \end{aligned}$$

Thus, (3.8) is the unique solution so that

$$f_0(\xi,\lambda) \sim e^{i\xi\lambda},$$

as claimed. □

Having these tools at our disposal, we proceed with our investigation of the Jost solutions. To this end, instead of the Volterra equation (2.13) we will work with the following representation of the solutions of (2.9):

**Lemma 3.5.** *For any $\xi > 0$, $\lambda > 0$,*

$$f_+(\xi,\lambda) = f_0(\xi,\lambda) + \int_\xi^\infty G_0(\xi,\eta;\lambda)V_1(\eta)f_+(\eta,\lambda)\,d\eta \tag{3.9}$$

*with $V_1$ as in (3.7), $f_0$ as in (3.8) and*

$$G_0(\xi,\eta;\lambda) = [\overline{f_0(\xi,\lambda)}f_0(\eta,\lambda)) - f_0(\xi,\lambda)\overline{f_0(\eta,\lambda)}](2i\lambda)^{-1}. \tag{3.10}$$

*For any small $\lambda > 0$ and $1 < \xi < \lambda^{-1}$,*

$$|G_0(\xi,\eta;\lambda)| \lesssim (\xi\eta)^{\frac12}|\log\lambda|^2\chi_{[\xi<\eta<\lambda^{-1}]} + (\xi/\lambda)^{\frac12}|\log\lambda|\chi_{[\eta>\lambda^{-1}]} \tag{3.11}$$

*Proof.* Simply observe that $G_0$ is the Green's function of our problem relative to $\mathcal{H}_0$. Indeed,

$$\begin{aligned} G_0(\xi,\xi;\lambda) &= 0,\\ \partial_\xi G_0(\xi,\eta;\lambda)|_{\eta=\xi} &= 1,\\ \mathcal{H}_0 G_0(\cdot,\eta;\lambda) &= \lambda^2 G_0(\cdot,\eta;\lambda). \end{aligned}$$

Here we have used that $W(f_0(\cdot,\lambda),\overline{f_0(\cdot,\lambda)}) = -2i\lambda$ which can be seen by computing the Wronskian at $\xi=\infty$. In conclusion,

$$\mathcal{H}_0 f_+(\xi,\lambda) = \lambda^2\left[f_0(\xi,\lambda) + \int_\xi^\infty G_0(\xi,\eta;\lambda)V_1(\eta)f_+(\eta,\lambda)\,d\eta\right] - V_1(\xi)f_+(\xi,\lambda)$$

or equivalently,

$$\mathcal{H}f_+(\cdot,\lambda) = \lambda^2 f_+(\cdot,\lambda).$$

Finally, observe that for $\xi > \lambda^{-1}$ fixed,

$$\sup_{\eta>\xi}|G_0(\xi,\eta;\lambda)| \lesssim \lambda^{-1}.$$

By the Volterra iteration discussed above, this implies that $|f_+(\xi,\lambda) - f_0(\xi,\lambda)| \lesssim \lambda^{-1}\xi^{-2}$. In particular,

$$f_+(\xi,\lambda) \sim e^{i\lambda\xi} \qquad \text{as } \xi\to\infty$$

For the estimate (3.11), recall the asymptotic bounds

$$H_0^{(+)}(x) = 1 + O_{\mathbb{R}}(x^2) + \frac{2}{\pi}i\log x + i\varkappa + iO_{\mathbb{R}}(x^2\log x) \tag{3.12}$$

as $x\to 0$ where $\varkappa$ is some real constant, see [1]. Moreover, $|H_0^{(+)}(x)| \lesssim x^{-\frac12}$ for all $x > 1$. Hence,

$$\begin{aligned} |G_0(\xi,\eta;\lambda)| &\lesssim (\xi\eta)^{\frac12}|H_0^{(+)}(\lambda\xi)||H_0^{(+)}(\lambda\eta)|\\ &\lesssim (\xi\eta)^{\frac12}|\log(\lambda\xi)|\Big(|\log(\lambda\eta)|\chi_{[\eta\lambda<1]} + (\eta\lambda)^{-\frac12}\chi_{[\eta\lambda\ge1]}\Big) \end{aligned}$$

which implies (3.11). □

Estimating the oscillatory integrals will require understanding $\partial_\lambda^k\partial_\xi^\ell f_\pm(\xi,\lambda)$, for $0\le k+\ell\le 2$, $W(\lambda)$, $W'(\lambda)$ and thus $a_\pm(\lambda)$, $b_\pm(\lambda)$, $a'_\pm(\lambda)$ and $b'_\pm(\lambda)$. To obtain asymptotic expansions for all these functions, we need to know the asymptotic behavior of $u_j(\xi)$, and thereafter that of $\partial_\lambda^k\partial_\xi^\ell u_j(\xi,\lambda)$, for $j=1$, 2 and $0\le k+\ell\le 2$.

To study the asymptotic behavior of the $u_j(\xi,\lambda)$, we use (3.3). Setting $h_j(\xi,\lambda) := \frac{u_j(\xi,\lambda)}{u_j(\xi)}$, for $\xi>0$ we obtain the integral equations

$$h_0(\xi,\lambda) = 1 + \frac{\lambda^2}{u_0(\xi)}\int_0^\xi [u_1(\xi)u_0^2(\eta) - u_0(\xi)u_1(\eta)u_0(\eta)]h_0(\eta,\lambda)\,d\eta, \tag{3.13}$$

$$h_1(\xi,\lambda) = 1 + \frac{\lambda^2}{u_1(\xi)}\int_0^\xi [u_1(\xi)u_0(\eta)u_1(\eta) - u_0(\xi)u_1^2(\eta)]h_1(\eta,\lambda)\,d\eta \tag{3.14}$$

from (3.3). The first iterates of (3.13) and (3.14) are controlled by the following lemma. The $O(\cdot)$ terms appearing here will be differentiated later, for now we only control their size.

**Corollary 3.6.** *As $\xi\to\infty$,*

$$u_1(\xi)\int_0^\xi u_0^2(\eta)\,d\eta - u_0(\xi)\int_0^\xi u_1u_0(\eta)\,d\eta = \frac{1}{4}2^{-1/4}\xi^{5/2} + O(\xi^{3/2}\log\xi) \tag{3.15}$$

$$\begin{aligned} u_1(\xi)\int_0^\xi u_0u_1(\eta)\,d\eta - u_0(\xi)\int_0^\xi u_1^2(\eta)\,d\eta = {} & \frac{1}{4}2^{1/4}\xi^{5/2}\log\xi \\ & + c_3\xi^{5/2} + O(\xi^{\frac{3}{2}}\log\xi) \end{aligned} \tag{3.16}$$

*where $c_3\in\mathbb{R}$ is some constant.*

*Proof.* By the asymptotic expressions for $u_0$ and $u_1$,

$$\begin{aligned} \int_0^\xi u_0^2(\eta)\,d\eta &= 2^{-1/2}\int_0^\xi \eta\left(1 - \frac{c_\infty}{\langle\eta\rangle} + O(\langle\eta\rangle^{-2})\right)d\eta \\ &= 2^{-1/2}\left(\frac{1}{2}\xi^2 - c_\infty\xi + O(\log\xi)\right) \\ \int_0^\xi u_0(\eta)u_1(\eta)\,d\eta &= \int_0^\xi \eta\left(1 - \frac{c_\infty}{\langle\eta\rangle} + O(\langle\eta\rangle^{-2})\right)\left(\log\eta + c_2 + O(\langle\eta\rangle^{-1})\right)d\eta \\ &= \frac{1}{2}\xi^2\log\xi + \frac{1}{2}\left(c_2 - \frac{1}{2}\right)\xi^2 + O(\xi\log\xi). \end{aligned}$$

Thus,

$$\begin{aligned} (3.15) &= 2^{-1/4}\xi^{1/2}(\log\xi + c_2 + O(\xi^{-1}\log\xi))\left(\frac{1}{2}\xi^2 + O(\xi)\right) \\ &\quad - 2^{-1/4}\xi^{1/2}(1 + O(\xi^{-1}))\left(\frac{1}{2}\xi^2\log\xi + \frac{1}{2}\left(c_2 - \frac{1}{2}\right)\xi^2 + O(\xi\log\xi)\right) \\ &= 2^{-1/4}\xi^{1/2}\left[\frac{1}{4}\xi^2 + O(\xi\log\xi)\right] \end{aligned}$$

Next, compute

$$\begin{aligned} \int_0^\xi u_1^2(\eta)\,d\eta &= \sqrt{2}\int_0^\xi \eta(\log^2\eta + 2c_2\log\eta + O(\langle\eta\rangle^{-1}\log\eta))(1 + O(\langle\eta\rangle^{-1}))\,d\eta \\ &= \sqrt{2}\left(\frac{1}{2}\xi^2\log^2\xi + (2c_2 - 1)\int_0^\xi \eta\log\eta\,d\eta + O(\xi\log^2\xi)\right) \\ &= \sqrt{2}\left(\frac{1}{2}\xi^2\log^2\xi + \frac{2c_2-1}{2}\xi^2\log\xi - \frac{2c_2-1}{4}\xi^2 + O(\xi\log^2\xi)\right) \end{aligned}$$

Thus, (3.16) equals

$$
\begin{aligned}
&2^{1/4}\xi^{1/2}(\log\xi + c_2 + O(\xi^{-1}))(1+O(\xi^{-1}))\left(\frac{1}{2}\xi^2\log\xi + \frac{1}{2}\left(c_2-\frac{1}{2}\right)\xi^2 + O(\xi\log\xi)\right)\\
&\quad - 2^{1/4}\xi^{1/2}(1+O(\xi^{-1}))\left(\frac{1}{2}\xi^2\log^2\xi + \frac{2c_2-1}{2}\xi^2\log\xi - \frac{2c_2-1}{4}\xi^2 + O(\xi\log^2\xi)\right)\\
&= 2^{1/4}\xi^{1/2}\Big\{\frac{1}{2}\xi^2\log^2\xi + \frac{2c_2-\frac{1}{2}}{2}\xi^2\log\xi + O(\xi\log^2\xi) + \frac{c_2}{2}\left(c_2-\frac{1}{2}\right)\xi^2\\
&\quad - \frac{1}{2}\xi^2\log^2\xi - \frac{2c_2-1}{2}\xi^2\log\xi + \frac{2c_2-1}{4}\xi^2\Big\}
\end{aligned}
$$

which finally reduces to

$$2^{1/4}\sqrt{\xi}\left(\frac{1}{4}\xi^2\log\xi + 2^{-1/4}c_3\xi^2 + O(\xi\log\xi)\right)$$

as claimed. □

Thus a Volterra iteration and the preceding yields the following result for the $u_j(\xi,\lambda)$'s. The importance of Corollary 3.7 lies with the fact that we do not lose $\log\xi$ factors in the $O(\cdot)$-terms as such factors would destroy the dispersive estimate. It is easy to see that carrying out the Volterra iteration crudely, by putting absolute values inside the integrals, leads to such $\log\xi$ losses. Therefore, we actually need to compute the Volterra iterates in (2.14) explicitly (for the version (∗∗)) .

**Corollary 3.7.** *In the range* $1 \ll \xi \lesssim \lambda^{-1}$, $j=0,1$,

$$
\begin{aligned}
&(3.17) && u_j(\xi,\lambda) = u_j(\xi)(1+O((\xi\lambda)^2))\\
& && \partial_\xi u_j(\xi,\lambda) = u_j'(\xi)(1+O((\xi\lambda)^2))\\
&(3.18) && \partial_\lambda u_0(\xi,\lambda) = \frac{1}{2}2^{-1/4}\lambda(\xi^{5/2} + O(\xi^{3/2}\log\xi))(1+O((\xi\lambda)^2))\\
& && \partial_\lambda u_1(\xi,\lambda) = \frac{1}{2}2^{1/4}\lambda(\xi^{5/2}\log\xi + c_3\xi^{5/2} + O(\xi^{3/2}\log\xi))(1+O((\xi\lambda)^2))\\
&(3.19) && \partial^2_{\lambda\xi} u_0(\xi,\lambda) = \frac{5}{4}2^{-1/4}\lambda(\xi^{3/2} + O(\xi^{1/2}\log\xi))(1+O((\xi\lambda)^2))\\
& && \partial^2_{\lambda\xi} u_1(\xi,\lambda) = \frac{5}{4}2^{1/4}\lambda(\xi^{3/2}\log\xi + \frac{2}{5}\xi^{3/2}\\
& && \qquad + c_3\xi^{3/2} + O(\xi^{1/2}\log\xi))(1+O((\xi\lambda)^2))
\end{aligned}
$$

*If* $|\xi| \lesssim 1$, *then* $|u_j(\xi,\lambda)| \lesssim 1$, $|\partial_\lambda u_j(\xi,\lambda)| + |\partial^2_{\lambda\xi} u_j(\xi,\lambda)| \lesssim \lambda$ *for* $j=0,1$.

*Proof.* We sketch the proof of this somewhat computational lemma, for the function $u_1(\xi,\lambda)$ since the argument for $u_0(\xi,\lambda)$ is completely analogous and in fact easier. The proof of the first equality in (3.17) is based on the Volterra integral equation (3.14)

$$h_1(\xi,\lambda) = 1 + \lambda^2\int_0^\xi \Big[\frac{u_1(\xi)u_0(\eta)u_1(\eta) - u_0(\xi)u_1^2(\eta)}{u_1(\xi)}\Big]h_1(\eta,\lambda)\,d\eta \qquad (3.20)$$

and its derivatives in both $\xi$ and $\lambda$ and the Volterra iteration, for which we also need to use Corollary 3.6. The iteration will produce a solution which is given by

$$h_1(\xi,\lambda) = 1 + \sum_{n=1}^{\infty} \lambda^{2n} \int_0^{\xi}\int_0^{\xi_1}\dots\int_0^{\xi_{n-1}} \frac{u_1(\xi)u_0(\xi_1)u_1(\xi_1) - u_0(\xi)u_1^2(\xi_1)}{u_1(\xi)}\cdots$$
$$\frac{u_1(\xi_{n-1})u_0(\xi_n)u_1(\xi_n) - u_0(\xi_{n-1})u_1^2(\xi_n)}{u_1(\xi_{n-1})}\,d\,\xi_n\dots d\,\xi_1 =$$
$$1 + \lambda^2 \int_0^{\xi} \frac{u_1(\xi)u_0(\xi_1)u_1(\xi_1) - u_0(\xi)u_1^2(\xi_1)}{u_1(\xi)}d\,\xi_1 +$$
$$\lambda^4 \int_0^{\xi}\int_0^{\xi_1} \frac{u_1(\xi)u_0(\xi_1)u_1(\xi_1) - u_0(\xi)u_1^2(\xi_1)}{u_1(\xi)}\,\frac{u_1(\xi_1)u_0(\xi_2)u_1(\xi_2) - u_0(\xi_1)u_1^2(\xi_2)}{u_1(\xi_1)}\,d\,\xi_2\,d\,\xi_1 + \cdots$$

Therefore, (3.16) and the equalities

$$u_0(\xi) = 2^{-1/4}\xi^{1/2}\left(1 - \frac{c_\infty}{2\xi} + O(\xi^{-2})\right)$$
$$u_1(\xi) = 2^{1/4}\xi^{1/2}\left(1 - \frac{c_\infty}{2\xi} + O(\xi^{-2})\right)\left(\log\xi + c_2 + O(\xi^{-1})\right)$$

yield

$$\begin{aligned}
h_1(\xi,\lambda) =& 1 + \frac{\lambda^2}{u_1(\xi)}\Big(\frac{1}{4}2^{1/4}\xi^{5/2}\log\xi + c_3\xi^{5/2} + O(\xi^{\frac{3}{2}}\log\xi)\Big) \\
& + \lambda^4\Big\{\int_0^{\xi} u_0(\xi_1)\Big[\frac{1}{4}2^{1/4}\xi_1^{5/2}\log\xi_1 + c_3\xi_1^{5/2} + O(\xi_1^{\frac{3}{2}}\log\xi_1)\Big]\,d\xi_1 - \\
& \frac{u_0(\xi)}{u_1(\xi)}\int_0^{\xi} u_1(\xi_1)\Big[\frac{1}{4}2^{1/4}\xi_1^{5/2}\log\xi_1 + c_3\xi_1^{5/2} + O(\xi_1^{\frac{3}{2}}\log\xi_1)\Big]\,d\xi_1\Big\} + \cdots \\
& = 1 + O(\lambda^2\xi^2),
\end{aligned}$$

since we are assuming that $1 \ll \xi \lesssim \lambda^{-1}$. The point to notice here is that terms involving $\xi^4\log\xi$ (the leading orders) after the integration cancel. Furthermore, we obtain the usual $n!$ gain from the Volterra iteration, see Lemma 2.4, from repeated integration of powers rather than from symmetry considerations. Hence $u_1(\xi,\lambda) = u_1(\xi)(1+O(\lambda^2\xi^2))$ in that range. To deal with the derivatives, it is more convenient to directly differentiate the integral equation (3.3) for $u_1(\xi,\lambda)$ with respect to $\xi$ and/or $\lambda$, which yields, respectively,

$$\partial_\xi u_1(\xi,\lambda) = \partial_\xi u_1(\xi) + \lambda^2\int_0^{\xi}[\partial_\xi u_1(\xi)u_0(\eta) - u_1(\eta)\partial_\xi u_0(\xi)]u_1(\eta,\lambda)d\eta \tag{3.21}$$

$$\begin{aligned}\partial_\lambda u_1(\xi,\lambda) = 2\lambda&\int_0^{\xi}[u_1(\xi)u_0(\eta) - u_1(\eta)u_0(\xi)]u_1(\eta,\lambda)d\eta \\ &+ \lambda^2\int_0^{\xi}[u_1(\xi)u_0(\eta) - u_1(\eta)u_0(\xi)]\partial_\lambda u_1(\eta,\lambda)d\eta,\end{aligned} \tag{3.22}$$

as well as

$$\partial^2_{\lambda\,\xi} u_1(\xi,\lambda) = 2\lambda \int_0^\xi [\partial_\xi u_1(\xi) u_0(\eta) - u_1(\eta)\partial_\xi u_0(\xi)] u_1(\eta,\lambda)\,d\eta$$
$$+ \lambda^2 \int_0^\xi [\partial_\xi u_1(\xi) u_0(\eta) - u_1(\eta)\partial_\xi u_0(\xi)] \partial_\lambda u_1(\eta,\lambda)\,d\eta. \tag{3.23}$$

In dealing with (3.21), we simply plug in the information from the first equality of (3.17) and calculate the resulting integral. For (3.22), we observe that by (3.16) the term

$$2\lambda \int_0^\xi [u_1(\xi)u_0(\eta) - u_1(\eta)u_0(\xi)] u_1(\eta,\lambda)\,d\eta$$

is equal to $\lambda(\frac{1}{2} 2^{1/4}\xi^{5/2}\log\xi + 2c_3\xi^{5/2} + O(\xi^{3/2}\log\xi))$. Therefore to solve (3.22), one needs to run the Volterra iteration with this expression as the first iterate. The treatment of (3.23) is similar to that of (3.22) and we skip the details. The case of $|\xi| \lesssim 1$ is left to the reader. □

We now turn to $f_\pm(\xi,\lambda)$ as well as $a_\pm$, $b_\pm(\lambda)$.

**Lemma 3.8.** *If $\lambda > 0$ is small, and $|\log\lambda|^2 \le \xi \ll \lambda^{-1}$, then*

$$f_+(\xi,\lambda) = f_0(\xi,\lambda) + O(\xi^{-1/2}\lambda^{\frac{1}{2}-\varepsilon})$$

*with $\varepsilon > 0$ arbitrary.*

*Proof.* Let

$$m(x) := \sqrt{x}\,|\log x|\,\chi_{[0<x<1]} + \chi_{[x>1]}$$

Then, in view of the asymptotic behavior of $H_0^{(+)}$,

$$|f_0(\xi,\lambda)| \lesssim m(\xi\lambda)$$

and thus also

$$|G_0(\xi,\eta;\lambda)| \lesssim \lambda^{-1}\, m(\xi\lambda) m(\eta\lambda)$$

We claim that also

$$|f_+(\xi,\lambda)| \lesssim m(\xi\lambda) \tag{3.24}$$

With $g(x;\lambda) := f_+(\xi,\lambda)/m(\xi\lambda)$, we obtain the integral inequality

$$g(\xi,\lambda) \le C + C\int_\xi^\infty \lambda^{-1}|V_1(\eta)| m(\eta\lambda)^2 g(\eta,\lambda)\,d\eta$$

for some absolute constant $C$. Since by our assumption on $\xi$,

$$\int_\xi^\infty \lambda^{-1}|V_1(\eta)|m(\eta\lambda)^2\,d\eta \lesssim \int_\xi^\infty \lambda^{-1}\eta^{-3} m(\eta\lambda)^2\,d\eta \lesssim \xi^{-1}|\log\lambda|^2 + \lambda \lesssim 1,$$

the claim follows from Lemma 2.4. We observed above that, see (3.11),

$$|G_0(\xi,\eta;\lambda)| \lesssim \sqrt{\xi\eta}\,|\log\lambda|^2\,\chi_{[\xi<\eta<\lambda^{-1}]} + \sqrt{\frac{\xi}{\lambda}}\,|\log\lambda|\,\chi_{[\eta>\lambda^{-1}]}$$

Thus integrating and taking $1 \ll \xi \ll \lambda^{-1}$ into account, we obtain from (3.24) that

$$\Big|\int_\xi^\infty G_0(\xi,\eta;\lambda)V_1(\eta)f_+(\eta,\lambda)\,d\eta\Big| \lesssim \int_\xi^{\lambda^{-1}} \sqrt{\xi\eta}\,|\log\lambda|^2\,\eta^{-3}\sqrt{\eta\lambda}\,|\log\lambda|\,d\eta$$
$$+ \int_{\lambda^{-1}}^\infty \sqrt{\frac{\xi}{\lambda}}\,|\log\lambda|\,\eta^{-3}\,d\eta \lesssim \xi^{-1/2}\lambda^{\frac{1}{2}-\varepsilon},$$

as claimed. □

We can now state our asymptotic expansion of $a_+$ and $b_+$. In what follows, $O(\cdot)$ terms are complex-valued unless stated to the contrary (which will be denoted by $O_{\mathbb{R}}(\cdot)$).

**Lemma 3.9.** *With $\varepsilon > 0$ arbitrary, small, and fixed,*

$$\begin{aligned} a_+(\lambda) &= 2^{1/4} c_0 \sqrt{\lambda}(1 + ic_1 \log\lambda + ic_3) + O(\lambda^{1-\varepsilon}) \\ b_+(\lambda) &= i2^{-1/4} c_0 c_1 \sqrt{\lambda} + O(\lambda^{1-\varepsilon}), \end{aligned} \tag{3.25}$$

*as $\lambda \to 0+$, where $c_0 = \sqrt{\frac{\pi}{2}} e^{i\frac{\pi}{4}}$, $c_1 = \frac{2}{\pi}$, and $c_3$ is some real constant.*

*Proof.* By Corollary 3.3 we have $a_+(\lambda) = f_+(\xi,\lambda)u_1'(\xi,\lambda) - f_+'(\xi,\lambda)u_1(\xi,\lambda)$. Hence Lemma 3.8 and Corollary 3.7 applied to $\xi = \lambda^{-1/2}$ yield,

$$\begin{aligned} c_0^{-1} 2^{1/4} a_+ &= \sqrt{\lambda\xi}\, H_0(\xi\lambda) \frac{1}{2} \xi^{-1/2} (\log\xi + c_2 + 2) \\ &\quad - \left( \frac{1}{2} \xi^{-1/2} \sqrt{\lambda}\, H_0(\xi\lambda) + \sqrt{\xi\lambda}\, H_0'(\xi\lambda)\lambda \right) \xi^{1/2} (\log\xi + c_2) + O(\lambda^{1-\varepsilon}) \\ &= \sqrt{\lambda}\, H_0(\xi\lambda) - \sqrt{\xi\lambda}\, \frac{ic_1}{\xi} \sqrt{\xi} (\log\xi + c_2) + O(\lambda^{1-\varepsilon}) \\ &= \sqrt{\lambda}(1 + ic_1 \log(\xi\lambda) + i\varkappa - ic_1\log\xi - ic_1c_2) + O(\lambda^{1-\varepsilon}) \\ &= \sqrt{\lambda}(1 + ic_1\log\lambda + ic_3) + O(\lambda^{1-\varepsilon}), \end{aligned}$$

as claimed. Note that $c_3 = \varkappa - c_1c_2$. Similarly,

$$\begin{aligned} -c_0^{-1} 2^{\frac{1}{4}} b_+ &= \sqrt{\lambda\xi}\, H_0(\xi\lambda) \frac{1}{2} \xi^{-1/2} - \xi^{1/2} \left( \frac{1}{2} \xi^{-1/2} \sqrt{\lambda}\, H_0(\lambda\xi) + \sqrt{\xi\lambda}\, H_0'(\xi\lambda)\lambda \right) \\ &\qquad\qquad + O(\lambda^{1-\varepsilon}) \\ &= -\xi\sqrt{\lambda}\, \frac{ic_1}{\xi\lambda}\, \lambda + O(\lambda^{1-\varepsilon}) = -ic_1\sqrt{\lambda} + O(\lambda^{1-\varepsilon}), \end{aligned}$$

and the lemma follows. □

Using the expressions for $a_+$ and $b_+$ above, we obtain the following

**Corollary 3.10.** *Let $\lambda > 0$ be small. Then*

$$f_+(\xi,\lambda) = c_0 \sqrt{\lambda\langle\xi\rangle} \left( 1 + ic_1 \log(\lambda\langle\xi\rangle) + ic_4 + O(\lambda^{\frac{1}{2}-\varepsilon}) + O(\langle\xi\rangle^{-1}\log\langle\xi\rangle) \right) \tag{3.26}$$

*for $0 < \xi < \lambda^{-1}$, whereas for $-\lambda^{-1} < \xi < 0$,*

$$f_+(\xi,\lambda) = c_0 \sqrt{\lambda\langle\xi\rangle} \left( 1 + ic_1 \log(\lambda\langle\xi\rangle^{-1}) + ic_5 + O(\lambda^{\frac{1}{2}-\varepsilon}) + O(\langle\xi\rangle^{-1}\log\langle\xi\rangle) \right) \tag{3.27}$$

*Here $c_1$ is as above and $c_4, c_5$ are real constants.*

*Proof.* This follows by inserting our asymptotic expansions for $a_+(\lambda)$, $b_+(\lambda)$, $u_0(\xi,\lambda)$, and $u_1(\xi,\lambda)$ into (3.5). □

We also need some information about certain partial derivatives of $f_+(\xi,\lambda)$. This is provided by

**Lemma 3.11.** *For $\lambda > 0$ small and $|\log \lambda|^2 \le \xi \ll \lambda^{-1}$ we have*

$$\begin{aligned}
\partial_\xi f_+(\xi,\lambda) &= \partial_\xi f_0(\xi,\lambda) + O(\xi^{-3/2}\lambda^{\frac12-\varepsilon}) \\
\partial_\lambda f_+(\xi,\lambda) &= \partial_\lambda f_0(\xi,\lambda) + O(\xi^{-1/2}\lambda^{-\frac12-\varepsilon}) \\
\partial^2_{\xi\lambda} f_+(\xi,\lambda) &= \partial^2_{\xi\lambda} f_0(\xi,\lambda) + O(\xi^{-3/2}\lambda^{-\frac12-\varepsilon})
\end{aligned}$$

*with $\varepsilon > 0$ arbitrary.*

*Proof.* This follows by taking derivatives in Lemma 3.8. □

To be able to carry out the analysis, one also needs to understand the derivative of the Wronskian. To that end we have

**Corollary 3.12.** *Then, with $\varepsilon > 0$ arbitrary but fixed,*

$$\begin{aligned}
a_+'(\lambda) &= \frac12 2^{1/4} c_0 \lambda^{-1/2}(1 + ic_3 + 2ic_1 + ic_1\log\lambda) + O(\lambda^{-\varepsilon}) \\
b_+'(\lambda) &= \frac{i}{2} 2^{-1/4} c_0 c_1 \lambda^{-1/2} + O(\lambda^{-\varepsilon})
\end{aligned} \tag{3.28}$$

*as $\lambda \to 0+$.*

*Proof.* In view of the preceding,

$$\begin{aligned}
a_+'(\lambda) &= W(\partial_\lambda f_+, u_1) + W(f_+, \partial_\lambda u_1) \\
&= W(\partial_\lambda f_0, u_1) + W(f_0, \partial_\lambda u_1) + O(\lambda^{-\varepsilon}) \\
&= \partial_\lambda[c_0\sqrt{\lambda\xi}H_0(\lambda\xi)]\left(\frac12\xi^{-1/2}(\log\xi + c_2) + \xi^{-1/2}\right)2^{1/4} \\
&\quad - \partial^2_{\lambda\xi}[c_0\sqrt{\lambda\xi}H_0(\lambda\xi)]\xi^{1/2}(\log\xi + c_2)\cdot 2^{1/4} \\
&\quad + c_0\sqrt{\lambda\xi}H_0(\lambda\xi)\cdot\frac54\cdot 2^{1/4}\lambda\left(\xi^{3/2}\log\xi + \left(\frac25 + c_3\right)\xi^{3/2}\right) \\
&\quad - c_0\partial_\xi[\sqrt{\lambda\xi}H_0(\lambda\xi)]\frac12 2^{1/4}\lambda(\xi^{5/2}\log\xi + c_3\xi^{5/2}) + O(\lambda^{-\varepsilon}).
\end{aligned} \tag{3.29}$$

Evaluating at $\xi = \lambda^{-1/2}$, one obtains that the third and fourth terms in (3.29) are $O(\lambda^{\frac12-\varepsilon})$, and thus error terms. Thus,

$$\begin{aligned}
2^{-1/4}c_0^{-1}a_+'(\lambda) &= \left(\frac12\lambda^{-1/2}(1 + ic_1\log(\lambda\xi) + i\varkappa) + ic_1\lambda^{-1/2}\right)\left(\frac12(c_2 + \log\xi) + 1\right) \\
&\quad - \left(\frac14\lambda^{-1/2}(1 + ic_1\log(\lambda\xi) + i\varkappa) + ic_1\lambda^{-1/2}\right)(\log\xi + c_2) + O(\lambda^{-\varepsilon}) \\
&= \frac12\lambda^{-1/2}(1 + ic_1\log(\lambda\xi) + i\varkappa) + ic_1\lambda^{-1/2}
\end{aligned}$$

which further simplifies to

$$\begin{aligned}
&\quad - \frac{ic_1}{2}\lambda^{-1/2}(\log\xi + c_2) + O(\lambda^{-\varepsilon}) \\
&= \frac12\lambda^{-1/2}(1 + ic_1\log\lambda + i\varkappa + 2ic_1 - ic_1c_2) + O(\lambda^{-\varepsilon}) \\
&= \frac12\lambda^{-1/2}(1 + ic_3 + 2ic_1 + ic_1\log\lambda) + O(\lambda^{-\varepsilon}).
\end{aligned}$$

Similarly,

$$\begin{aligned} 2^{-1/4}c_0^{-1}b_+'(\lambda) &= \frac{1}{2}\left(\frac{1}{2}\lambda^{-1/2}(1+ic_1\log(\lambda\xi)+i\varkappa)+ic_1\lambda^{-1/2}\right) \\ &\quad -\left(\frac{1}{4}\lambda^{-1/2}(1+ic_1\log(\lambda\xi)+i\varkappa)+ic_1\lambda^{-1/2}\right)+O(\lambda^{-\varepsilon}) \\ &= -\frac{1}{2}ic_1\lambda^{-1/2}+O(\lambda^{-\varepsilon}), \end{aligned}$$

as claimed. □

*Remark* 3.13. Recall that this analysis was carried out assuming that $\mathcal{M}$ is conical on the right. If $\mathcal{M}$ is conical on the left, then the same analysis applies. In fact, if $\mathcal{M}$ is symmetric, i.e., $r(x) = r(-x)$, then by Corollary 3.3 $a_-(\lambda) = a_+(\lambda)$ and $b_-(\lambda) = -b_+(\lambda)$. If it is not symmetric but still conical at both ends, then these relations still hold for the asymptotic expansions. i.e.,

$$\begin{aligned} a_-(\lambda) &= 2^{1/4}c_0\sqrt{\lambda}(1+ic_1\log\lambda+ic_3)+O(\lambda^{1-\varepsilon}) \\ b_-(\lambda) &= -i2^{-1/4}c_0c_1\sqrt{\lambda}+O(\lambda^{1-\varepsilon}), \end{aligned}$$

as $\lambda \to 0+$ where $c_0$ etc. are as in Lemma 3.9. The same of course applies to $a_-'$ and $b_-'$.

We end the perturbative analysis with a description of the oscillatory behavior of $f_+(\xi,\lambda)$ for $\lambda\xi > 1$.

**Lemma 3.14.** *Let $m_+(\xi,\lambda) := e^{-i\lambda\xi}f_+(\xi,\lambda)$. Then, provided $\lambda > 0$ is small and $\lambda\xi > 1$,*

$$\begin{aligned} |m_+(\xi,\lambda)-1| &\lesssim (\lambda\xi)^{-1} \\ |\partial_\lambda m_+(\xi,\lambda)| &\lesssim \lambda^{-2}\xi^{-1} \end{aligned} \tag{3.30}$$

*Proof.* From (3.9), and with $m_0(\xi,\lambda) = e^{-i\lambda\xi}f_0(\xi,\lambda)$,

$$m_+(\xi,\lambda) = m_0(\xi,\lambda) + \int_\xi^\infty \widetilde{G}_0(\xi,\eta;\lambda)V_1(\eta)m_+(\eta,\lambda)\,d\eta \tag{3.31}$$

where

$$\widetilde{G}_0(\xi,\eta;\lambda) = \frac{m_0(\xi,\lambda)\overline{m_0(\eta,\lambda)} - e^{-2i(\xi-\eta)\lambda}\overline{m_0(\xi,\lambda)}m_0(\eta,\lambda)}{-2i\lambda} \tag{3.32}$$

Now, by asymptotic properties of the Hankel functions,

$$m_0(\xi,\lambda) = 1 + O((\xi\lambda)^{-1})$$

where the $O$-term behaves like a symbol.[1] Inserting this bound into (3.32) yields

$$|\widetilde{G}_0(\xi,\eta;\lambda)| \lesssim \eta$$

provided $\eta > \xi > \lambda^{-1}$. Thus, from (3.31),

$$|m_+(\xi,\lambda) - m_0(\xi,\lambda)| \lesssim \xi^{-1}$$

and thus, for all $\xi\lambda > 1$,

$$|m_+(\xi,\lambda) - 1| \lesssim (\xi\lambda)^{-1}$$

[1] In fact, $m_0(\xi,\lambda) = 1 + O_{\mathbb{R}}((\xi\lambda)^{-2}) + iO_{\mathbb{R}}((\xi\lambda)^{-1})$.

as claimed.
Next, one checks that for $\eta > \xi > \lambda^{-1}$,

$$|\partial_\lambda \widetilde{G}_0(\xi,\eta;\lambda)| \lesssim \frac{\eta}{\lambda}.$$

Thus, for all $\lambda\xi > 1$,

$$\begin{aligned}|\partial_\lambda m_+(\xi,\lambda)| \lesssim& \lambda^{-2}\xi^{-1} + \int_\xi^\infty |\partial_\lambda \widetilde{G}_0(\xi,\eta;\lambda)|\eta^{-3}\,d\eta + \int_\xi^\infty \eta^{-2}|\partial_\lambda m_+(\eta\lambda)|\,d\eta \\ \lesssim& \lambda^{-2}\xi^{-1} + \lambda^{-1}\xi^{-1} + \int_\xi^\infty \eta^{-2}|\partial_\lambda m_+(\eta,\lambda)|\,d\eta \lesssim \lambda^{-1}(\lambda\xi)^{-1},\end{aligned}$$

as claimed. □

3.2. **The Wronksian $W(\lambda)$ for conical ends, $d=1, n=0$.** In view of our asymptotic analysis of $a_\pm$ and $b_\pm$ and an explicit expression for the Wronskian $W(\lambda)$ in terms of these functions, see Corollary 3.3, we now derive the following important fact.

**Corollary 3.15.** *As $\lambda \to 0+$,*

$$\begin{aligned}W(\lambda) &= 2\lambda\left(1 + ic_3 + i\frac{2}{\pi}\log\lambda\right) + O(\lambda^{\frac{3}{2}-\varepsilon}) \\ W'(\lambda) &= 2\left(1 + ic_3 + i\frac{2}{\pi} + i\frac{2}{\pi}\log\lambda\right) + O(\lambda^{\frac{1}{2}-\varepsilon})\end{aligned}$$

*with $\varepsilon > 0$ arbitrary.*

*Proof.* Follows immediately from

$$W(\lambda) = (a_- b_+ - a_+ b_-)(\lambda)$$

and (3.25), (3.28). See Remark 3.13. □

## 4. The oscillatory integral estimates for $d = 1, n = 0$

We now commence with proving the main oscillatory integral estimate (2.16) and (2.17) for small energies. Thus, let $\chi$ be a smooth cut-off function to small energies, i.e., $\chi(\lambda) = 1$ for small $|\lambda|$ and $\chi$ vanishes outside a small interval around zero. In addition, we introduce the smooth cut-off functions $\chi_{[|\xi\lambda|<1]}$ and $\chi_{[|\xi\lambda|>1]}$ which form a partition of unity adapted to these intervals.

**Lemma 4.1.** *For all $t > 0$*

$$\sup_{\xi,\xi'}\left|\int_0^\infty e^{it\lambda^2}\lambda\frac{\chi(\lambda;\xi,\xi')}{\langle\xi\rangle\langle\xi'\rangle)^{1/2}}\mathrm{Im}\left[\frac{f_+(\xi,\lambda)f_-(\xi',\lambda)}{W(\lambda)}\right]\,d\lambda\right| \lesssim \langle t\rangle^{-1} \tag{4.1}$$

$$\sup_{\xi,\xi'}\left|\int_0^\infty e^{\pm it\lambda}\lambda\frac{\chi(\lambda;\xi,\xi')}{(\langle\xi\rangle\langle\xi'\rangle)^{1/2}}\mathrm{Im}\left[\frac{f_+(\xi,\lambda)f_-(\xi',\lambda)}{W(\lambda)}\right]\,d\lambda\right| \lesssim \langle t\rangle^{-1} \tag{4.2}$$

*where $\chi(\lambda;\xi,\xi') := \chi(\lambda)\chi_{[|\xi\lambda|<1,|\xi'\lambda|<1]}$.*

*Proof.* We shall first assume for simplicity that $\mathcal{M}$ is symmetric, i.e., $r(x) = r(-x)$. The general case will be discussed at the end of the proof. We first observe the

following:

$$\begin{aligned}
&\mathrm{Im}\left[\frac{f_+(\xi,\lambda)f_-(\xi',\lambda)}{W(\lambda)}\right] \\
&\quad= \mathrm{Im}\left[\frac{(a_+(\lambda)u_0(\xi,\lambda)+b_+(\lambda)u_1(\xi,\lambda))(a_+(\lambda)u_0(\xi',\lambda)-b_+(\lambda)u_1(\xi',\lambda))}{-2a_+b_+(\lambda)}\right] \\
&\quad= -\frac{1}{2}\mathrm{Im}\left(\frac{a_+}{b_+}(\lambda)\right)u_0(\xi,\lambda)u_0(\xi',\lambda)+\frac{1}{2}\mathrm{Im}\left(\frac{b_+}{a_+}(\lambda)\right)u_1(\xi,\lambda)u_1(\xi',\lambda).
\end{aligned}$$

Further, by (3.25), with $\varepsilon > 0$ arbitrary but fixed,

$$\begin{aligned}
-\frac{1}{2}\mathrm{Im}\left(\frac{a_+}{b_+}(\lambda)\right) &= \frac{\pi}{2\sqrt{2}}\mathrm{Re}\left[\frac{1+ic_1\log\lambda+ic_3+O(\lambda^{\frac{1}{2}-\varepsilon})}{1+O(\lambda^{\frac{1}{2}-\varepsilon})}\right] \\
&= O_{\mathbb{R}}(\lambda^{\frac{1}{2}-\varepsilon})+\frac{\pi}{2\sqrt{2}}
\end{aligned}$$

and by Corollary 3.12, the $O$-term can be formally differentiated, i.e.,

$$\frac{d}{d\lambda}\left\{-\frac{1}{2}\mathrm{Im}\left(\frac{a_+}{b_+}(\lambda)\right)\right\} = O_{\mathbb{R}}(\lambda^{-\frac{1}{2}-\varepsilon}).$$

Similarly,

$$\frac{1}{2}\mathrm{Im}\left(\frac{b_+}{a_+}(\lambda)\right) = -\frac{\sqrt{2}}{\pi}\,\frac{1}{1+(c_3+c_1\log\lambda)^2}+O_{\mathbb{R}}(\lambda^{\frac{1}{2}-\varepsilon})$$

which can again be formally differentiated.
By the estimates of Corollary 3.7, provided $|\xi\lambda|+|\xi'\lambda|<1$,

$$\begin{aligned}
|u_0(\xi,\lambda)u_0(\xi',\lambda)| &\lesssim \sqrt{\langle\xi\rangle\langle\xi'\rangle} \\
|\partial_\lambda[u_0(\xi,\lambda)u_0(\xi',\lambda)]| &\lesssim \lambda\left(\langle\xi\rangle^{5/2}\langle\xi'\rangle^{1/2}+\langle\xi'\rangle^{5/2}\langle\xi\rangle^{1/2}\right) \\
&\lesssim \lambda\sqrt{\langle\xi\rangle\langle\xi'\rangle}(\langle\xi\rangle^2+\langle\xi'\rangle^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
|u_1(\xi,\lambda)u_1(\xi',\lambda)| &\lesssim \sqrt{\langle\xi\rangle\langle\xi'\rangle}\log(2+\langle\xi\rangle)\log(2+\langle\xi'\rangle) \\
|\partial_\lambda[u_1(\xi,\lambda)u_1(\xi',\lambda)]| &\lesssim \lambda\sqrt{\langle\xi\rangle\langle\xi'\rangle}(\langle\xi\rangle^2+\langle\xi'\rangle^2)\log(2+\langle\xi\rangle)\log(2+\langle\xi'\rangle)
\end{aligned}$$

Passing absolute values inside (4.1) and (4.2) shows that these expressions are dominated by

$$\begin{aligned}
&\int_0^\infty\left|\chi(\xi,\xi';\lambda)(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\mathrm{Im}\left(\frac{a_+}{b_+}(\lambda)\right)u_0(\xi,\lambda)u_0(\xi',\lambda)\right|d\lambda \\
&\quad+\int_0^\infty\left|\chi(\xi,\xi';\lambda)(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\mathrm{Im}\left(\frac{b_+}{a_+}(\lambda)\right)u_1(\xi,\lambda)u_1(\xi',\lambda)\right|d\lambda
\end{aligned}\tag{4.3}$$

which is bounded by an absolute constant. To obtain decay in $t$, we integrate by parts. Integrating by parts in (4.1) yields that it is dominated by

$$\begin{aligned}
&t^{-1}\int_0^\infty\left|\partial_\lambda\Big[\chi(\xi,\xi';\lambda)(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\mathrm{Im}\left(\frac{a_+}{b_+}(\lambda)\right)u_0(\xi,\lambda)u_0(\xi',\lambda)\Big]\right|d\lambda \\
&\quad+t^{-1}\int_0^\infty\left|\partial_\lambda\Big[\chi(\xi,\xi';\lambda)(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\mathrm{Im}\left(\frac{b_+}{a_+}(\lambda)\right)u_1(\xi,\lambda)u_1(\xi',\lambda)\Big]\right|d\lambda
\end{aligned}\tag{4.4}$$

Using the bounds we derived above these expressions can be seen to be $\lesssim t^{-1}$ and (4.1) holds. For (4.2) we write $e^{it\lambda} = (it)^{-1}\partial_\lambda e^{it\lambda}$ and integrate by parts; this yields that the left-hand side of (4.2) is dominated by the exact same terms as in (4.4) (in fact, with an extra $\lambda$).

If $\mathcal{M}$ is not symmetric, then the asymptotics of the previous section allow for the following conclusion (in very much the same way as in the symmetric case):

$$\begin{aligned}\mathrm{Im}\left[\frac{f_+(\xi,\lambda)f_-(\xi',\lambda)}{W(\lambda)}\right] &= \big(\gamma_0 + O_{\mathbb{R}}(\lambda^{\frac12-\varepsilon})\big)u_0(\xi,\lambda)u_0(\xi',\lambda)\\ &\quad + \big(\frac{\gamma_1}{1+(c_3+c_1\log\lambda)^2} + O_{\mathbb{R}}(\lambda^{\frac12-\varepsilon})\big)u_1(\xi,\lambda)u_1(\xi',\lambda)\\ &\quad + O_{\mathbb{R}}(\lambda^{\frac12-\varepsilon})(u_0(\xi,\lambda)u_1(\xi',\lambda) + u_1(\xi,\lambda)u_0(\xi',\lambda))\end{aligned}$$

where $\gamma_0, \gamma_1$ are nonzero real constants (in fact, the same as in the symmetric case). With this representation in hand, the oscillatory integrals are estimated exactly as in the symmetric case. $\square$

Next, we consider the case $|\xi\lambda| > 1$ and $|\xi'\lambda| > 1$. With the convention that $f_\pm(\xi,-\lambda) = \overline{f_\pm(\xi,\lambda)}$ we can remove the imaginary part in (2.16) and integrate $\lambda$ over the whole axis. We shall follow this convention hence forth. To estimate the oscillatory integrals, we shall repeatedly use the following version of stationary phase, see Lemma 2 in [18] for the proof.

**Lemma 4.2.** *Let $\phi(0) = \phi'(0) = 0$ and $1 \le \phi'' \le C$. Then*

$$\left|\int_{-\infty}^{\infty} e^{it\phi(x)}a(x)\,dx\right| \lesssim \delta^2\left\{\int \frac{|a(x)|}{\delta^2+|x|^2}\,dx + \int_{|x|>\delta}\frac{|a'(x)|}{|x|}\,dx\right\} \tag{4.5}$$

*where $\delta = t^{-1/2}$.*

Using Lemma 4.2 we can prove the following:

**Lemma 4.3.** *With $\chi(\lambda;\xi,\xi') = \chi(\lambda)\chi_{[|\xi\lambda|>1,|\xi'\lambda|>1]}$,*

$$\sup_{\xi>0>\xi'}\left|\int_{-\infty}^{\infty} e^{it\lambda^2}\lambda\chi(\lambda;\xi,\xi')(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\frac{f_+(\xi,\lambda)f_-(\xi',\lambda)}{W(\lambda)}\,d\lambda\right| \lesssim t^{-1} \tag{4.6}$$

$$\sup_{\xi>0>\xi'}\left|\int_{-\infty}^{\infty} e^{\pm it\lambda}\lambda\chi(\lambda;\xi,\xi')(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\frac{f_+(\xi,\lambda)f_-(\xi',\lambda)}{W(\lambda)}\,d\lambda\right| \lesssim t^{-\frac12} \tag{4.7}$$

*for all $t > 0$.*

*Proof.* Writing $f_+(\xi,\lambda) = e^{i\xi\lambda}m_+(\xi,\lambda)$, $f_-(\xi,\lambda) = e^{-i\xi\lambda}m_-(\xi,\lambda)$ as in Lemma 3.14, we express (4.6) in the form

$$\left|\int_{-\infty}^{\infty} e^{it\phi(\lambda)}a(\lambda)\,d\lambda\right| \lesssim t^{-1} \tag{4.8}$$

where $\xi > 0 > \xi'$ are fixed, $\phi(\lambda) := \lambda^2 + \frac{\lambda}{t}(\xi-\xi')$, and

$$a(\lambda) = \lambda\chi(\lambda)\chi_{[|\xi\lambda|>1,|\xi'\lambda|>1]}(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\frac{m_+(\xi,\lambda)m_-(\xi',\lambda)}{W(\lambda)}.$$

Let $\lambda_0 = -\frac{\xi-\xi'}{2t}$. We have the bounds

$$|a(\lambda)| \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{\frac{-1}{2}}\chi(\lambda)\chi_{[|\xi\lambda|>1,|\xi'\lambda|>1]}. \tag{4.9}$$

By Corollary 3.15, for small $|\lambda|$

$$\Big|\Big(\frac{\lambda}{W(\lambda)}\Big)'\Big| \lesssim \frac{1}{|\lambda|(\log|\lambda|)^2}$$

and by Lemma 3.14, for $|\xi\lambda|>1$, $|\xi'\lambda|>1$,

$$|\partial_\lambda[m_+(\xi,\lambda)m_-(\xi',\lambda)]| \lesssim \lambda^{-2}(\xi^{-1}+|\xi'|^{-1}).$$

Hence,

$$|a'(\lambda)| \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\chi(\lambda)\chi_{[|\xi\lambda|>1,|\xi'\lambda|>1]}\left\{\frac{|\lambda|^{-1}}{|\log\lambda|^2}+\lambda^{-2}(\xi^{-1}+|\xi'|^{-1})\right\}. \tag{4.10}$$

We will need to consider three cases in order to prove (4.8) via (4.5), depending on where $\lambda_0$ falls relative to the support of $a$.

Case 1: $|\lambda_0|\lesssim 1$, $|\lambda_0|\gtrsim |\xi|^{-1}+|\xi'|^{-1}$.

Note that the second inequality here implies that

$$\frac{\xi+|\xi'|}{t}\gtrsim\frac{\xi+|\xi'|}{\xi|\xi'|}\quad\text{or}\quad 1\gtrsim\frac{t}{\xi|\xi'|}.$$

Furthermore, we remark that $a\equiv 0$ unless $\xi\gtrsim 1$ and $|\xi'|\gtrsim 1$.
Starting with the first integral on the right-hand side of (4.5) we conclude from (4.9) that

$$\int\frac{|a(\lambda)|}{|\lambda-\lambda_0|^2+\delta^2}\,d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-1/2}t^{1/2}\lesssim 1.$$

From the second integral we obtain from (4.10) that

$$\begin{aligned}\int\limits_{|\lambda-\lambda_0|>\delta}\frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda &\lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\delta^{-1}\int\frac{\chi(\lambda)\,d\lambda}{|\lambda|(\log|\lambda|)^2}\\ &\quad+(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}(\langle\xi\rangle^{-1}+\langle\xi'\rangle^{-1})\,\delta^{-1}\int\limits_{\lambda>\xi^{-1}+|\xi'|^{-1}}\frac{d\lambda}{\lambda^2}\\ &\lesssim\sqrt{\frac{t}{\langle\xi\rangle\langle\xi'\rangle}}\lesssim 1.\end{aligned}$$

Case 2: $|\lambda_0|\lesssim 1$, $|\lambda_0|\ll\langle\xi\rangle^{-1}+\langle\xi'\rangle^{-1}$.

Then $|\lambda-\lambda_0|\sim|\lambda|$ on the support of $a$, which implies that

$$\int\frac{|a(\lambda)|}{|\lambda-\lambda_0|^2+t^{-1}}\,d\lambda\lesssim(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\int\limits_{\lambda>\xi^{-1}+|\xi'|^{-1}}\frac{d\lambda}{\lambda^2}\lesssim\frac{\sqrt{\xi|\xi'|}}{\xi+|\xi'|}\lesssim 1,$$

and also

$$\begin{aligned}&\int\limits_{|\lambda-\lambda_0|>\delta}\frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda\\ &\lesssim(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\Big(\int\limits_{\lambda>\xi^{-1}+|\xi'|^{-1}}\frac{d\lambda}{\lambda^2(\log|\lambda|)^2}+\int\limits_{\lambda>\xi^{-1}+|\xi'|^{-1}}\frac{d\lambda}{\lambda^3}(\xi^{-1}+|\xi'|^{-1})\Big)\\ &\lesssim\frac{\sqrt{\xi|\xi'|}}{\xi+|\xi'|}\lesssim 1.\end{aligned}$$

Case 3: $|\lambda_0| >> 1$, $|\lambda_0| \gtrsim \xi^{-1} + |\xi'|^{-1}$.

In this case, $|\lambda - \lambda_0| \sim |\lambda_0| >> 1$. Thus,

$$\int \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2 + t^{-1}}\, d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-1/2} \frac{1}{\lambda_0^2 + t^{-1}} \lesssim 1$$

as well as, see (4.10),

$$\int\limits_{|\lambda-\lambda_0|>\delta} \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\, d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\lambda_0^{-1} \int \frac{\chi(\lambda)}{|\lambda|(\log|\lambda|)^2}\, d\lambda$$

$$+ \int \frac{1}{\lambda^2}\chi_{[|\lambda|>\xi^{-1}+|\xi'|^{-1}]} \frac{d\lambda}{\lambda_0}\, \frac{\xi+|\xi'|}{(\xi|\xi'|)^{3/2}} \lesssim 1,$$

and (4.6) is proved.

Integrating by parts shows that (4.7) is dominated by

$$(1+|t\pm(\xi-\xi')|)^{-1}\int (|a(\lambda)|+|a'(\lambda)|)\, d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}(1+|t\pm(\xi-\xi')|)^{-1}$$

which is $\lesssim t^{-\frac12}$ and the lemma is proved. $\square$

Now we turn to the estimate of the oscillatory integral for the case $|\xi\lambda| > 1$ and $|\xi'\lambda| < 1$.

**Lemma 4.4.** *Let $\chi(\lambda;\xi,\xi') = \chi_{[|\xi\lambda|>1, |\xi'\lambda|<1]}\chi(\lambda)$. Then*

$$\sup_{\xi>0>\xi'} \left| (\langle\xi\rangle\langle\xi'\rangle)^{-1/2} \int_{-\infty}^{\infty} e^{it\lambda^2} \frac{\lambda\chi(\lambda;\xi,\xi')}{W(\lambda)} f_+(\xi,\lambda) f_-(\xi',\lambda)\, d\lambda \right| \lesssim t^{-1} \tag{4.11}$$

$$\sup_{\xi>0>\xi'} \left| (\langle\xi\rangle\langle\xi'\rangle)^{-1/2} \int_{-\infty}^{\infty} e^{\pm it\lambda} \frac{\lambda\chi(\lambda;\xi,\xi')}{W(\lambda)} f_+(\xi,\lambda) f_-(\xi',\lambda)\, d\lambda \right| \lesssim t^{-\frac12} \tag{4.12}$$

*for all $t > 0$ and similarly with $\chi_{[|\xi\lambda|<1, |\xi'\lambda|>1]}$.*

*Proof.* As before, we write $f_+(\xi,\lambda) = e^{i\xi\lambda} m_+(\xi,\lambda)$. But because of $|\xi'\lambda| < 1$ we use the representation

$$f_-(\xi',\lambda) = a_-(\lambda)u_0(\xi',\lambda) + b_-(\lambda)u_1(\xi',\lambda).$$

In particular,

$$|f_-(\xi',\lambda)| \lesssim \sqrt{|\lambda|\langle\xi'\rangle}\,\big|\log|\lambda|\big|.$$

Moreover, from (3.18) and (3.28),

$$|\partial_\lambda f_-(\xi',\lambda)| \lesssim \langle\xi'\rangle^{1/2}|\lambda|^{-1/2}\big|\log|\lambda|\big|$$

provided $|\xi'\lambda| < 1$. We apply (4.5) with $\phi(\lambda) = \lambda^2 + \frac{\xi}{t}\lambda$ and

$$a(\lambda) = \frac{\lambda\chi(\lambda)}{W(\lambda)}(\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\chi_{[|\xi\lambda|>1,|\xi'\lambda|<1]} m_+(\xi,\lambda) f_-(\xi',\lambda).$$

By the preceding,

$$|a(\lambda)| \lesssim \frac{|\lambda|^{1/2}}{\sqrt{\langle\xi\rangle}}\chi(\lambda)\chi_{[|\xi\lambda|>1,|\xi'\lambda|<1]} \tag{4.13}$$

and

$$|a'(\lambda)| \lesssim (|\lambda|\langle\xi\rangle)^{-1/2}\chi(\lambda)\chi_{[|\xi\lambda|>1,|\xi'\lambda|<1]}. \tag{4.14}$$

Case 1: $|\lambda_0| \lesssim 1$, $|\xi\lambda_0| \gtrsim 1$.

Note in particular $|\xi| \gtrsim 1$. Here $\lambda_0 = -\frac{\xi}{2t}$. By (4.13),

$$\begin{aligned}\int \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2+t^{-1}}\,d\lambda &\lesssim \langle\xi\rangle^{-1/2}\int \frac{\sqrt{|\lambda|}}{|\lambda-\lambda_0|^2+t^{-1}}\,d\lambda \\ &\lesssim \langle\xi\rangle^{-1/2}|\lambda_0|^{1/2}\int \frac{d\lambda}{|\lambda-\lambda_0|^2+t^{-1}} + \langle\xi\rangle^{-1/2}\int \frac{|\lambda|^{1/2}}{|\lambda|^2+t^{-1}}\,d\lambda \\ &\lesssim \langle\xi\rangle^{-1/2}t^{1/2}\left(\frac{\xi}{t}\right)^{1/2} + \langle\xi\rangle^{-1/2}t^{1/4} \lesssim 1\end{aligned}$$

Here we used that $|\xi\lambda_0| = \frac{\xi^2}{2t} \gtrsim 1$.
Next, write via (4.14)

$$\int_{|\lambda-\lambda_0|>\delta} \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda \lesssim \langle\xi\rangle^{-\frac{1}{2}}\int_{|\lambda-\lambda_0|>\delta} \frac{1}{|\lambda|^{\frac{1}{2}}|\lambda-\lambda_0|}\chi_{[|\xi\lambda|>1,|\xi'\lambda|<1]}\,d\lambda. \tag{4.15}$$

Distinguish the cases $\frac{1}{10}|\lambda| > |\lambda-\lambda_0|$ and $\frac{1}{10}|\lambda| \le |\lambda-\lambda_0|$ in the integral on the right-hand side. This yields

$$\begin{aligned}(4.15) &\lesssim \langle\xi\rangle^{-1/2}\int_{|\lambda-\lambda_0|>\delta} \frac{d\lambda}{|\lambda-\lambda_0|^{3/2}} + \langle\xi\rangle^{-1/2}\int_{|\lambda|\lesssim|\lambda_0|} \frac{d\lambda}{|\lambda|^{1/2}}|\lambda_0|^{-1} \\ &\quad + \langle\xi\rangle^{-1/2}\int_{|\lambda|>|\lambda_0|} \frac{d\lambda}{|\lambda|^{3/2}} \\ &\lesssim \langle\xi\rangle^{-1/2}\delta^{-1/2} + \langle\xi\rangle^{-1/2}|\lambda_0|^{-1/2} \lesssim \left(\frac{t}{\xi^2}\right)^{1/4} + |\xi\lambda_0|^{-1/2} \lesssim 1.\end{aligned}$$

<u>Case 2:</u> $|\lambda_0| \lesssim 1$, $|\xi\lambda_0| \ll 1$

In that case, $|\lambda-\lambda_0| \sim |\lambda|$ on the support of $a$. Consequently,

$$\int \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2+t^{-1}}\,d\lambda \lesssim \langle\xi\rangle^{-\frac{1}{2}}\int_{|\xi|^{-1}}^{\infty} |\lambda|^{-\frac{3}{2}}\,d\lambda \lesssim 1.$$

Moreover,

$$\int_{|\lambda-\lambda_0|>\delta} \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda \lesssim \int_{|\xi|^{-1}}^{\infty} \frac{(|\lambda|\langle\xi\rangle)^{-\frac{1}{2}}}{|\lambda|}\,d\lambda \lesssim 1.$$

<u>Case 3:</u> $|\lambda_0| >> 1$.

In that case, $|\lambda-\lambda_0| \sim |\lambda_0|$ on $\mathrm{supp}(a)$. Since $|a(\lambda)| \lesssim 1$ by (4.13), it follows that

$$\int \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2+t^{-1}}\,d\lambda \lesssim 1.$$

Similarly, since $|a'(\lambda)| \lesssim (\xi|\lambda|)^{-\frac{1}{2}}$, it follows that

$$\int_{|\lambda-\lambda_0|>\delta} \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda \lesssim \int \frac{(|\lambda|\langle\xi\rangle)^{-\frac{1}{2}}}{|\lambda_0|}\chi(\lambda)\,d\lambda \lesssim 1.$$

This proves (4.11).

To prove (4.12), we integrate by parts to obtain the upper bound

$$(1+|t\pm\xi|)^{-1}\int (|a(\lambda)|+|a'(\lambda)|)\,d\lambda \lesssim (1+|t\pm\xi|)^{-1}\xi^{-\frac{1}{2}} \lesssim t^{-\frac{1}{2}}$$

and the lemma is proved. The other case $\chi_{[|\xi\lambda|<1,|\xi'\lambda|>1]}$ is treated in an analogous fashion. $\square$

The remaining cases for the small energy part of (2.16) are $\xi > \xi' > |\lambda|^{-1}$ and $\xi' < \xi < -|\lambda|^{-1}$. By symmetry it will suffice to treat the former case. As usual, we need to consider reflection and transmission coefficients, therefore we write

$$f_-(\xi,\lambda) = \alpha_-(\lambda) f_+(\xi,\lambda) + \beta_-(\lambda)\overline{f_+(\xi,\lambda)}. \tag{4.16}$$

Then, with $W(\lambda) = W(f_+(\cdot,\lambda), f_-(\cdot,\lambda))$,

$$W(\lambda) = \beta_-(\lambda) W(f_+(\cdot,\lambda), \overline{f_+(\cdot,\lambda)}) = -2i\lambda\beta_-(\lambda)$$

and

$$\begin{aligned} W(f_-(\cdot,\lambda), \overline{f_+(\cdot,\lambda)}) &= \alpha_-(\lambda) W(f_+(\cdot,\lambda), \overline{f_+(\cdot,\lambda)}) \\ &= -2i\lambda\alpha_-(\lambda). \end{aligned}$$

Thus, when $\lambda > 0$ is small,

$$\beta_-(\lambda) = i\left(1 + ic_3 + i\frac{2}{\pi}\log\lambda\right) + O(|\lambda|^{\frac12-\varepsilon}) \tag{4.17}$$

and

$$\begin{aligned} \alpha_-(\lambda) &= \frac{1}{-2i\lambda} W\left(a_+(\lambda)u_0(\cdot,\lambda) - b_+(\lambda)u_1(\cdot,\lambda), \overline{a_+}(\lambda)u_0(\cdot,\lambda) + \overline{b_+}(\lambda)u_1(\cdot,\lambda)\right) \\ &= \frac{1}{-2i\lambda}\left(a_+\overline{b_+}(\lambda) + \overline{a_+}(\lambda)b_+(\lambda)\right) \\ &= \frac{i}{\lambda}\mathrm{Re}(a_+\overline{b_+}(\lambda)) \\ &= \frac{i}{\lambda}\mathrm{Re}\left(-i|c_0|^2 c_1\lambda(1 + ic_1\log\lambda + ic_3) + O(\lambda^{\frac32-\varepsilon})\right) \\ &= i\left(\frac{2}{\pi}\log\lambda + c_3\right) + O(\lambda^{\frac12-\varepsilon}). \end{aligned} \tag{4.18}$$

In passing, we remark that $1 + |\alpha_-|^2 = |\beta_-|^2$. Finally, it follows from Corollary 3.12 that the $O$-terms can be differentiated once in $\lambda$; they then become $O(\lambda^{-\frac12-\varepsilon})$, $\varepsilon > 0$ arbitrary.

**Lemma 4.5.** *For any* $t > 0$

$$\sup_{\xi>\xi'>0}\left|(\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\int e^{it\lambda^2}\frac{\lambda\chi(\lambda)}{W(\lambda)}\chi_{[\xi'\lambda>1]} f_+(\xi,\lambda)f_-(\xi',\lambda)\,d\lambda\right| \lesssim t^{-1} \tag{4.19}$$

$$\sup_{\xi>\xi'>0}\left|(\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\int e^{\pm it\lambda}\frac{\lambda\chi(\lambda)}{W(\lambda)}\chi_{[\xi'\lambda>1]} f_+(\xi,\lambda)f_-(\xi',\lambda)\,d\lambda\right| \lesssim t^{-\frac12} \tag{4.20}$$

*and similarly for* $\sup_{\xi'<\xi<0}$ *and* $\chi_{[|\xi\lambda|>1]}$.

*Proof.* Using (4.16), we reduce (4.19) to two estimates:

$$\sup_{\xi>\xi'>0}\left|\int e^{it\lambda^2}e^{i\lambda(\xi+\xi')}\frac{\lambda\chi(\lambda)}{W(\lambda)}\frac{\chi_{[\xi'|\lambda|>1]}}{\sqrt{\langle\xi\rangle\langle\xi'\rangle}} m_+(\xi,\lambda)m_+(\xi',\lambda)\alpha_-(\lambda)\,d\lambda\right| \lesssim t^{-1} \tag{4.21}$$

and

$$\sup_{\xi>\xi'>0}\left|\int e^{it\lambda^2}e^{i\lambda(\xi-\xi')}\frac{\lambda\chi(\lambda)}{W(\lambda)}\frac{\chi_{[\xi'|\lambda|>1]}}{\sqrt{\langle\xi\rangle\langle\xi'\rangle}} m_+(\xi,\lambda)\overline{m_+(\xi',\lambda)}\beta_-(\lambda)\,d\lambda\right| \lesssim t^{-1} \tag{4.22}$$

We apply (4.5) to (4.21) with fixed $\xi > \xi' > 0$ and

$$\begin{aligned}
&\phi(\lambda) = \lambda^2 + \frac{\lambda}{t}(\xi+\xi'),\\
&a(\lambda) = (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\frac{\lambda\chi(\lambda)}{W(\lambda)}\chi_{[\xi'|\lambda|>1]}\alpha_-(\lambda)m_+(\xi,\lambda)m_+(\xi',\lambda).
\end{aligned}$$

Then from (4.18),

$$|a(\lambda)| \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\chi(\lambda)\chi_{[\xi'|\lambda|>1]} \tag{4.23}$$

and from our derivative bounds on $W$, $\alpha_-$, and $m_+(\xi,\lambda)$, see (3.30) for the latter, we conclude that

$$|a'(\lambda)| \lesssim |\lambda|^{-1}(\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\chi(\lambda)\chi_{[\xi'|\lambda|>1]}. \tag{4.24}$$

This bound will suffice for the Schrödinger evolution. For the wave evolution, we also need an integrable estimate on $|a'(\lambda)|$. It is

$$|a'(\lambda)| \lesssim |\lambda|^{-1}(\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\chi(\lambda)\chi_{[\xi'|\lambda|>1]}\Big(|\log\lambda|^{-2} + |\lambda\xi'|^{-1}\Big)$$

which one obtains by combining (4.18) with our asymptotic bound for $\frac{\lambda}{W(\lambda)}$ above.

<u>Case 1:</u> Suppose $|\lambda_0| \lesssim 1$ and $|\xi'\lambda_0| > 1$, where $\lambda_0 = -\frac{\xi+\xi'}{2t}$. Note $\xi > \xi' \gtrsim 1$.

Then

$$\int \frac{|a(\lambda)|}{|\lambda-\lambda_0|^2 + t^{-1}}\,d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\int\frac{d\lambda}{|\lambda-\lambda_0|^2+t^{-1}} \lesssim \sqrt{\frac{t}{\xi\xi'}} \lesssim 1$$

since $|\xi'\lambda_0| \sim \frac{\xi\xi'}{t} > 1$. As for the derivative term in (4.5), we infer from (4.24) that

$$\int_{|\lambda-\lambda_0|>\delta}\frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\int_{|\lambda-\lambda_0|>\delta}\frac{d\lambda}{|\lambda||\lambda-\lambda_0|}\chi_{[|\lambda\xi'|>1]} \tag{4.25}$$

Again, we need to distinguish between $|\lambda-\lambda_0| > \frac{1}{10}|\lambda_0|$ and $|\lambda-\lambda_0| < \frac{1}{10}|\lambda_0|$. Thus, since $\xi\xi' > t$,

$$\begin{aligned}
(4.25) &\lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\int_{1/\xi'}^\infty \frac{d\lambda}{\lambda^2} + (\langle\xi\rangle\langle\xi'\rangle)^{-1/2}|\lambda_0|^{-1}\log\Big(t^{1/2}|\lambda_0|\Big)\\
&\lesssim 1 + \frac{t^{\frac12}}{\xi}\log\left(\frac{\xi}{t^{1/2}}\right) \lesssim 1
\end{aligned}$$

since also $\xi^2 > t$.

<u>Case 2:</u> $|\lambda_0| \lesssim 1$, $|\lambda_0| \ll \frac{1}{\xi'}$.

Then $|\lambda-\lambda_0| \sim |\lambda|$ on the support of $a(\lambda)$. Hence,

$$\int\frac{|a(\lambda)|}{|\lambda-\lambda_0|^2+t^{-1}}\,d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\int_{1/\xi'}^\infty\frac{d\lambda}{\lambda^2} \lesssim \sqrt{\frac{\xi'}{\langle\xi\rangle}} < 1$$

and

$$\int_{|\lambda-\lambda_0|>\delta}\frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\int_{1/\xi'}^\infty\frac{d\lambda}{\lambda^2} < 1.$$

<u>Case 3:</u> $|\lambda_0| >> 1$, $|\lambda_0| \gtrsim \frac{1}{\xi'}$.

Then $|\lambda-\lambda_0| \sim |\lambda_0|$ on $\mathrm{supp}(a)$. Therefore, $|a(\lambda)| \lesssim 1$ implies that

$$\int\frac{|a(\lambda)|}{|\lambda-\lambda_0|^2+t^{-1}}\,d\lambda \lesssim 1$$

and

$$\int_{|\lambda-\lambda_0|>\delta} \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}}|\lambda_0|^{-1}\int_{\frac{1}{\langle\xi'\rangle}}^{1}\frac{d\lambda}{|\lambda|} \lesssim \frac{1}{|\lambda_0|}\frac{1}{\langle\xi'\rangle}\log\langle\xi'\rangle \lesssim 1.$$

This concludes the proof of (4.21). (4.22) is completely analogous and (4.19) follows.

As usual, integration by parts proves that (4.20) is dominated by

$$(1+|t\pm(\xi\pm\xi')|)^{-1}\int(|a(\lambda)|+|a'(\lambda)|)\,d\lambda \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}}(1+|t\pm(\xi\pm\xi')|)^{-1}$$

which is $\lesssim t^{-\frac{1}{2}}$.

Finally, the case of $\xi'<\xi<0$, $|\xi\lambda|>1$ follows from the case considered in this proof by a reflection around $\xi=0$. $\square$

We are done with the contributions of small $\lambda$ to the oscillatory integral (2.16) and (2.17). To conclude the proof of (1.5) for $d=1$ it suffices to prove the following statement. The wave equation will be treated separately, see Lemma 4.7.

**Lemma 4.6.** *For all $t>0$,*

$$\sup_{\xi>\xi'}\left|(\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{it\lambda^2}\frac{\lambda(1-\chi)(\lambda)}{W(\lambda)}f_+(\xi,\lambda)f_-(\xi',\lambda)\,d\lambda\right| \lesssim t^{-1}. \tag{4.26}$$

*Proof.* We observed above, see (4.16), that $W(\lambda)=-2i\lambda\beta_-(\lambda)$. Since $|\beta_-(\lambda)|\geq 1$, this implies that $|W(\lambda)|\geq 2|\lambda|$. In particular, $W(\lambda)\neq 0$ for every $\lambda\neq 0$. In order to prove (4.26), we will need to distinguish the cases $\xi>0>\xi'$, $\xi>\xi'>0$, and $0>\xi>\xi'$. By symmetry, it will suffice to consider the first two.

<u>Case 1:</u> $\xi>0>\xi'$.

In this case we need to prove that

$$\sup_{\xi>0>\xi'}\left|(\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}}\int e^{it[\lambda^2+\frac{\xi-\xi'}{t}\lambda]}\frac{\lambda(1-\chi)(\lambda)}{W(\lambda)}m_+(\xi,\lambda)m_-(\xi',\lambda)\,d\lambda\right| \lesssim t^{-1}. \tag{4.27}$$

Apply (4.5) with $\phi(\lambda)=\lambda^2+\frac{\xi-\xi'}{t}\lambda$ and

$$a(\lambda)=(\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}}\frac{\lambda(1-\chi)(\lambda)}{W(\lambda)}m_+(\xi,\lambda)m_-(\xi',\lambda).$$

Hence, with $\lambda_0=-\frac{\xi-\xi'}{2t}$,

$$\begin{aligned}(4.27) &\lesssim t^{-1}\left(\int\frac{|a(\lambda)|}{|\lambda-\lambda_0|^2+t^{-1}}\,d\lambda+\int\limits_{|\lambda-\lambda_0|>\delta}\frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\,d\lambda\right)\\ &=: t^{-1}(A+B).\end{aligned}$$

If $|\lambda_0|\ll 1$, then

$$A\lesssim ||a||_\infty \lesssim 1.$$

On the other hand, if $|\lambda_0|\gtrsim 1$, then $\xi+|\xi'|\gtrsim t$ so that

$$A\lesssim t^{\frac{1}{2}}||a||_\infty \lesssim t^{\frac{1}{2}}(\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}} \lesssim \sqrt{\frac{t}{\langle\xi\rangle\langle\xi'\rangle}} \lesssim 1.$$

Here we used that

$$\sup_{\xi}\sup_{|\lambda|\gtrsim 1}|m_{\pm}(\xi,\lambda)|\lesssim 1$$

which follows from the fact that

$$m_+(\xi,\lambda)=1+\int_{\xi}^{\infty}\frac{1-e^{-2i(\tilde{\xi}-\xi)\lambda}}{2i\lambda}V(\tilde{\xi})m_+(\tilde{\xi},\lambda)d\tilde{\xi} \tag{4.28}$$

with $V(\tilde{\xi}=O(\langle\tilde{\xi}\rangle^{-2})$. Moreover, from our assumptions on $r(x)$ we recall that

$$\left|\frac{d^{\ell}}{d\xi^{\ell}}V(\xi)\right|\lesssim\langle\xi\rangle^{-2-\ell},\quad\forall\,\ell\ge 0.$$

We shall need these bounds to estimate $B$ above. From (4.28), for $\xi\ge 0$

$$m_+(\xi,\lambda)=1+O(\lambda^{-1}\langle\xi\rangle^{-1})$$

as well as for $\xi\ge 0$

$$\partial_{\xi}^{j}m_+(\xi,\lambda)=O(\lambda^{-1}\langle\xi\rangle^{-1-j}),\qquad j=1,2 \tag{4.29}$$

$$\partial_{\lambda}m_+(\xi,\lambda)=O(\lambda^{-2}\langle\xi\rangle^{-1}) \tag{4.30}$$

$$\partial_{\lambda}\partial_{\xi}m_+(\xi,\lambda)=O(\lambda^{-2}\langle\xi\rangle^{-2}) \tag{4.31}$$

To verify (4.29), one checks that

$$\begin{aligned}\partial_{\xi}m_+(\xi,\lambda)&=\frac{1}{2i\lambda}\int_{\xi}^{\infty}[1-e^{2i(\xi-\tilde{\xi})\lambda}]V'(\tilde{\xi})m_+(\tilde{\xi},\lambda)\,d\tilde{\xi}\\&+\frac{1}{2i\lambda}\int_{\xi}^{\infty}[1-e^{2i(\xi-\tilde{\xi})\lambda}]V(\tilde{\xi})\partial_{\tilde{\xi}}m_+(\tilde{\xi},\lambda)\,d\tilde{\xi}.\end{aligned} \tag{4.32}$$

By our estimates on $V$, the integral on the right-hand side of (4.32) is $O(\lambda^{-1}\langle\xi\rangle^{-2})$ and (4.29) follows for $j=1$. For $j=2$ note that

$$\begin{aligned}\partial_{\xi}^{2}m_+(\xi,\lambda)&=\frac{1}{2i\lambda}\int_{\xi}^{\infty}[1-e^{2i(\xi-\tilde{\xi})\lambda}]\,V''(\tilde{\xi})m_+(\tilde{\xi},\lambda)\,d\tilde{\xi}\\&+\frac{1}{i\lambda}\int_{\xi}^{\infty}[1-e^{2i(\xi-\tilde{\xi})\lambda}]\,V'(\tilde{\xi})\partial_{\tilde{\xi}}m_+(\tilde{\xi},\lambda)\,d\tilde{\xi}\\&+\frac{1}{2i\lambda}\int_{\xi}^{\infty}[1-e^{2i(\xi-\tilde{\xi})\lambda}]V(\tilde{\xi})\partial_{\tilde{\xi}}^{2}m_+(\tilde{\xi},\lambda)\,d\tilde{\xi}.\end{aligned}$$

which again implies the desired bound. For (4.30) we compute

$$\begin{aligned}\partial_{\lambda}m_+(\xi,\lambda)=&-\int_{\xi}^{\infty}\frac{1-e^{2i(\xi-\tilde{\xi})\lambda}}{2i\lambda^2}V(\tilde{\xi})m_+(\tilde{\xi},\lambda)\,d\tilde{\xi}\\&+\frac{1}{2i\lambda^2}\int_{\xi}^{\infty}e^{2i(\xi-\tilde{\xi})\lambda}\partial_{\tilde{\xi}}\left[(\xi-\tilde{\xi})V(\tilde{\xi})m_+(\tilde{\xi},\lambda)\right]\,d\tilde{\xi}\\&+\int_{\xi}^{\infty}\frac{1-e^{2i(\xi-\tilde{\xi})\lambda}}{2i\lambda}V(\tilde{\xi})\partial_{\lambda}m_+(\tilde{\xi},\lambda)\,d\tilde{\xi}\end{aligned}$$

so that

$$\partial_{\lambda}m_+(\xi,\lambda)=O(\lambda^{-2}\langle\xi\rangle^{-1})$$

as claimed. Finally, compute

$$
\begin{aligned}
\partial^2_{\xi\lambda} m_+(\xi,\lambda) &= \frac{1}{\lambda}\int_\xi^\infty e^{2i(\xi-\tilde\xi)\lambda} V(\tilde\xi) m_+(\tilde\xi,\lambda)\,d\tilde\xi \\
&\quad + \frac{1}{2i\lambda^2} V(\xi) m_+(\xi,\lambda) + \frac{1}{2i\lambda}\int_\xi^\infty e^{2i(\xi-\tilde\xi)\lambda}\partial_{\tilde\xi}[(\xi-\tilde\xi)Vm_+(\tilde\xi,\lambda)]\,d\tilde\xi \\
&\quad + \frac{1}{2i\lambda^2}\int_\xi^\infty e^{2i(\xi-\tilde\xi)\lambda}\partial_{\tilde\xi}[V(\tilde\xi)m_+(\tilde\xi,\lambda)]\,d\tilde\xi \\
&\quad - \int_\xi^\infty e^{2i(\xi-\tilde\xi)\lambda} V(\tilde\xi)\partial_\lambda m_+(\tilde\xi,\lambda)\,d\tilde\xi
\end{aligned}
$$

Integrating by parts in the first and third terms, and using the previous bounds, yields the desired estimate. As a corollary, we obtain (take $\xi = 0$)

$$
\begin{aligned}
W(\lambda) &= W(f_+(\cdot,\lambda), f_-(\cdot,\lambda)) \\
&= m_+(\xi,\lambda)[m_-'(\xi,\lambda) - i\lambda m_-(\xi,\lambda)] - m_-(\xi,\lambda)[m_+'(\xi,\lambda) + i\lambda m_+(\xi,\lambda)] \\
&= -2i\lambda(1+O(\lambda^{-1})) + O(\lambda^{-1}) = -2i\lambda + O(1)
\end{aligned}
$$

with derivatives $W'(\lambda) = -2i + O(\lambda^{-1})$ as $|\lambda| \to \infty$.
Next, we estimate $B$. First, we conclude from our bounds on $W(\lambda)$ and $m_+(\xi,\lambda)$ as well as $m_-(\xi',\lambda)$ that

$$
|a'(\lambda)| \lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12}\chi_{[|\lambda|\gtrsim 1]}|\lambda|^{-2}.
$$

Let us first consider the case where $|\lambda_0| \gtrsim 1$. Then

$$
\begin{aligned}
B &\lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\int_{\left(\substack{|\lambda-\lambda_0|>\delta \\ |\lambda|\gtrsim 1}\right)} \frac{d\lambda}{|\lambda|^2|\lambda-\lambda_0|} \\
&\lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-1/2}\left\{\int_1^\infty \frac{d\lambda}{\lambda^3} + \frac{1}{|\lambda_0|^2}\int_{\frac{|\lambda_0|}{5}>|\lambda-\lambda_0|>\delta}\frac{d\lambda}{|\lambda-\lambda_0|}\right\} \\
&\lesssim 1 + \sqrt{\frac{t}{\langle\xi\rangle\langle\xi'\rangle}}\,\frac{1}{|\lambda_0|t^{1/2}}\log_+(\lambda_0 t^{1/2}) \lesssim 1
\end{aligned}
$$

Here we used that $\frac{t}{\langle\xi\rangle\langle\xi'\rangle} \lesssim 1$ which follows from $|\lambda_0| \gtrsim 1$. If $|\lambda_0| \ll 1$, then $|\lambda-\lambda_0| \sim |\lambda|$ on the support of $a$; thus $B \lesssim 1$ trivially. This finishes the case $\xi > 0 > \xi'$.

Case 2: To deal with the case $\xi > \xi' > 0$, we use (4.16). Thus,

$$
f_-(\xi',\lambda) = \alpha_-(\lambda) f_+(\xi',\lambda) + \beta_-(\lambda)\overline{f_+(\xi',\lambda)}
$$

where

$$
\begin{aligned}
\alpha_-(\lambda) &= \frac{W(f_-(\cdot,\lambda), \overline{f_+(\cdot,\lambda)})}{-2i\lambda} \\
\beta_-(\lambda) &= \frac{W(f_+(\cdot,\lambda), f_-(\cdot,\lambda))}{-2i\lambda} = \frac{W(\lambda)}{-2i\lambda}
\end{aligned}
$$

From our large $\lambda$ asymptotics of $W(\lambda)$ we deduce that

$$
\beta_-(\lambda) = 1 + O(\lambda^{-1}), \quad \beta_-'(\lambda) = O(\lambda^{-2}). \tag{4.33}
$$

For $\alpha_-(\lambda)$ we calculate, again at $\xi = 0$,

$$\begin{aligned} W(f_-(\cdot,\lambda), \overline{f_+(\cdot,\lambda)}) &= m_-(\xi,\lambda)(\overline{m}'_+(\xi,\lambda) - 2i\lambda\overline{m}_+(\xi,\lambda)) \\ &\quad - \overline{m}_+(\xi,\lambda)(m'_-(\xi,\lambda) - 2i\lambda m_-(\xi,\lambda)) \\ &= m_-(\xi,\lambda)\overline{m}'_+(\xi,\lambda) - m'_-(\xi,\lambda)\overline{m}_+(\xi,\lambda) \\ &= O(\lambda^{-1}) \end{aligned}$$

so that

$$\alpha_-(\lambda) = O(\lambda^{-2}), \quad \alpha'_-(\lambda) = O(\lambda^{-3}). \tag{4.34}$$

Thus, we are left with proving the two bounds

$$\sup_{\xi>\xi'>0}\left| \int_{-\infty}^{\infty} e^{it\lambda^2} e^{i\lambda(\xi+\xi')} \frac{\lambda(1-\chi(\lambda))}{W(\lambda)} \alpha_-(\lambda) \frac{m_+(\xi,\lambda)m_+(\xi',\lambda)}{\sqrt{\langle\xi\rangle\langle\xi'\rangle}}\, d\lambda \right| \lesssim t^{-1} \tag{4.35}$$

$$\sup_{\xi>\xi'>0}\left| \int_{-\infty}^{\infty} e^{it\lambda^2} e^{i\lambda(\xi-\xi')} \frac{\lambda(1-\chi(\lambda))}{W(\lambda)} \beta_-(\lambda) \frac{m_+(\xi,\lambda)\overline{m_+(\xi',\lambda)}}{\sqrt{\langle\xi\rangle\langle\xi'\rangle}}\, d\lambda \right| \lesssim t^{-1} \tag{4.36}$$

for any $t > 0$. This, however, follows by means of the exact same arguments which we use to prove (4.27). Note that in (4.35) the critical point of the phase is

$$\lambda_0 = -\frac{\xi+\xi'}{2t}$$

whereas in (4.36) it is $\lambda_0 = -\frac{\xi-\xi'}{2t}$. In either case it follows from $|\lambda_0| \gtrsim 1$ that $\xi \gtrsim t$. Hence we can indeed argue as in Case 1. This finishes the proof of the lemma, and thus also of Theorem 1.2. □

Now for the wave case. We will tacitly use some elements of the previous proof.

**Lemma 4.7.** *For all $t > 0$,*

$$\begin{aligned} &\left| \int_{-\infty}^{\xi} (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12} \int_{-\infty}^{\infty} e^{\pm it\lambda} \frac{\lambda(1-\chi)(\lambda)}{W(\lambda)} f_+(\xi,\lambda) f_-(\xi',\lambda)\, d\lambda\, \phi(\xi')\, d\xi' \right| \\ &\lesssim t^{-\frac12} \int \big(|\phi(\xi')| + |\phi'(\xi')|\big)\, d\xi'. \end{aligned} \tag{4.37}$$

*with a constant that does not depend on $\xi$.*

*Proof.* In order to prove (4.37), we will need to distinguish the cases $\xi > 0 > \xi'$, $\xi > \xi' > 0$, and $0 > \xi > \xi'$. By symmetry, it will suffice to consider the first two.

Case 1: $\xi > 0 > \xi'$.

Integrating by parts yields

$$\begin{aligned} &\left| (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12} \int e^{i\lambda(\pm t+\xi-\xi')} \frac{\lambda(1-\chi)(\lambda)}{W(\lambda)} m_+(\xi,\lambda) m_-(\xi',\lambda)\, d\lambda \right| \\ &\lesssim (\langle\xi\rangle\langle\xi'\rangle)^{-\frac12} |t \pm (\xi-\xi')|^{-1} \lesssim t^{-\frac12} \end{aligned}$$

provided $|t \pm (\xi - \xi')| \geq 1$. If this fails, then we need to integrate by parts in $\xi'$ to remove one factor of $\lambda$: since $\lambda e^{-i\xi'\lambda} = i\partial_{\xi'} e^{-i\xi'\lambda}$, it follows that

$$\int_{-\infty}^{\xi} \langle\xi\rangle^{-\frac{1}{2}} \langle\xi'\rangle^{-\frac{1}{2}} \int e^{i\lambda(\pm t+\xi-\xi')} \frac{\lambda(1-\chi)(\lambda)}{W(\lambda)} m_+(\xi,\lambda) m_-(\xi',\lambda)\, d\lambda\, \phi(\xi')\, d\xi' =$$

$$i\langle\xi\rangle^{-1} \int e^{\pm it\lambda} \frac{(1-\chi)(\lambda)}{W(\lambda)} m_+(\xi,\lambda) m_-(\xi,\lambda)\, d\lambda\, \phi(\xi)$$

$$- i \int_{-\infty}^{\xi} \langle\xi\rangle^{-\frac{1}{2}} \int e^{i\lambda(\pm t+\xi-\xi')} \frac{(1-\chi)(\lambda)}{W(\lambda)} m_+(\xi,\lambda) \partial_{\xi'} \big[\langle\xi'\rangle^{-\frac{1}{2}} m_-(\xi',\lambda)\, \phi(\xi')\big]\, d\lambda\, d\xi'$$

Denote the two expressions after the equality sign by $A$ and $B$, respectively. First, exploiting the cancelation due to $W(-\lambda) = -W(\lambda) + O(1)$ as $\lambda \to \infty$, we see that

$$\sup_{\xi>0>\xi'} \Big| \int e^{it\lambda} \frac{(1-\chi)(\lambda)}{W(\lambda)} m_+(\xi,\lambda) m_-(\xi,\lambda)\, d\lambda \Big| \lesssim 1$$

Furthermore, since $|\partial_\lambda \{ \frac{(1-\chi)(\lambda)}{W(\lambda)} m_+(\xi,\lambda) m_-(\xi,\lambda)\}| \lesssim \chi_{[|\lambda|\gtrsim 1]} |\lambda|^{-2}$, integrating by parts in $\lambda$ shows that the left-hand side is in fact $\lesssim t^{-1}$. Hence,

$$A \lesssim \langle t\rangle^{-1} \sup|\phi| \leq \langle t\rangle^{-1} \int (|\phi'(\xi')| + |\phi(\xi')|)\, d\xi'.$$

Second, by the same cancelation,

$$\begin{aligned} B &\lesssim \int (\langle\xi\rangle\langle\xi'\rangle)^{-\frac{1}{2}} (1 + |t \pm (\xi - \xi')|)^{-1} (|\phi'(\xi')| + |\phi(\xi')|)\, d\xi' \\ &\lesssim \langle t\rangle^{-1} \int (|\phi'(\xi')| + |\phi(\xi')|)\, d\xi'. \end{aligned}$$

which gives the desired bound as usual.

Case 2: $\xi > \xi' > 0$

In analogy with (4.35) and (4.36) we need to consider

$$\int_{-\infty}^{\infty} e^{it\lambda} e^{i\lambda(\xi+\xi')} \frac{\lambda(1-\chi(\lambda))}{W(\lambda)} \alpha_-(\lambda) \frac{m_+(\xi,\lambda) m_+(\xi',\lambda)}{\sqrt{\langle\xi\rangle\langle\xi'\rangle}}\, d\lambda, \tag{4.38}$$

$$\int_{-\infty}^{\infty} e^{it\lambda} e^{i\lambda(\xi-\xi')} \frac{\lambda(1-\chi(\lambda))}{W(\lambda)} \beta_-(\lambda) \frac{m_+(\xi,\lambda) \overline{m_+(\xi',\lambda)}}{\sqrt{\langle\xi\rangle\langle\xi'\rangle}}\, d\lambda. \tag{4.39}$$

The integral in (4.38) is $\lesssim \langle t\rangle^{-\frac{1}{2}}$ uniformly in $\xi, \xi'$ due to the decay of $\alpha_-$, see (4.34). On the other hand, the integral in (4.39) is not a bounded function in $\xi, \xi'$ due to the lack of decay in $\lambda$, see (4.33). Thus, we again need to redeem one power of $\lambda$ via a $\xi'$ differentiation, see above. $\square$

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